Universal Scaling Laws in Quantum Quench

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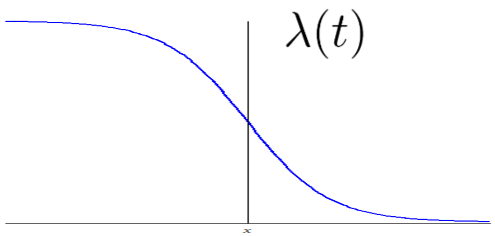
Excited States and Quantum Quench

- An efficient way to produce excited states in quantum many-body systems is to subject it to a quantum quench.
- Typically one can start with the ground state of an initial (*time independent*) Hamiltonian, and then introduce time dependence in a coupling so that at late times one gets another time-independent Hamiltonian.



- The (*Heisenberg picture*) state is then an excited state of the new Hamiltonian.

- This excited state then evolves non-trivially in time and we are typically interested in a variety of questions about the resulting non-equilibrium state.
- One set of questions relate to thermalization : how does the system reach a steady state, and in what sense is this state (approximately) thermal ?

- Another set of questions relate to universal properties of various quantities for situations where the quench involves the vicinity of critical points. These questions sometimes involve early time behavior – in the critical region.

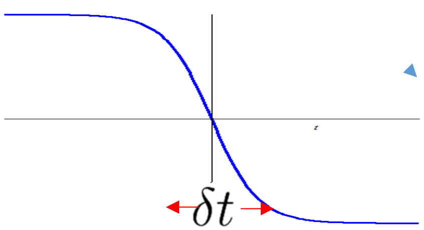
- An important aspect of this is the scaling of expectation values with the quench rate, the exponents being determined by the critical point.

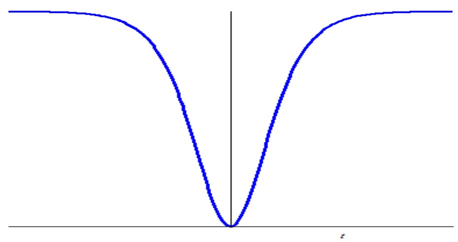
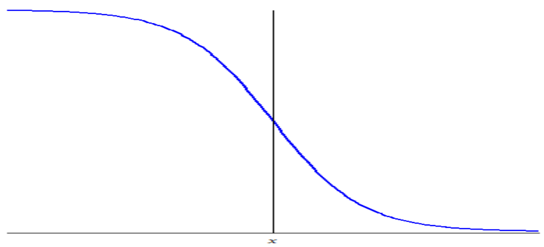
- In this talk I will explore some well known as well as some newly discovered scaling properties.

Scaling in Critical Quantum Quench

- Quantum quench involving critical points can be described by an action

$$S = S_{critical} - \int dt \int d^{d-1}x \ \lambda(t)\mathcal{O}(\vec{x}, t) \qquad \lambda(t) = \lambda_0 F(t/\delta t)$$



- Near the critical point the coupling has the behavior

$$\lambda(t) = \lambda_0 (t/\delta t)^r$$

Kibble Zurek Scaling

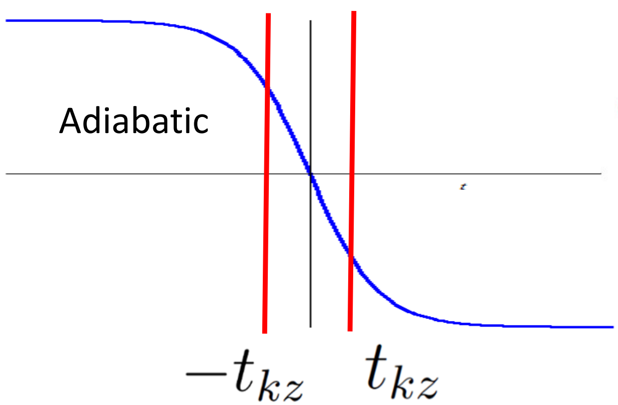
- When the quench is slow compared to the physical mass scales in the problem

$$\delta t \gg \lambda_0^{-1/(d-\Delta)}$$

- Kibble and Zurek proposed that after the time $-t_{kz}$ when adiabatic evolution breaks down the system becomes, to lowest approximation, diabatic.
- The instantaneous correlation length at $-t_{kz}$ is the only length scale in the critical region

$$t_{kz} \sim \left(\frac{\delta t}{\lambda_0^{1/r}}\right)^{\frac{z\nu}{z\nu+1}}$$

ν = Correlation length exponent
z = Dynamical critical exponent



$$\xi_{kz} = \xi(t_{kz}) = \left(\frac{\delta t}{\lambda_0^{1/r}}\right)^{\frac{\nu}{z\nu+1}}$$

- The instantaneous energy gap is given by

$$E_{gap}(t) \sim [\lambda_0(\frac{t}{\delta t})^r]^{z\nu}$$

- The time at which adiabaticity fails is given by Landau criterion

$$[\frac{1}{E_{gap}(t)^2}\frac{dE_{gap}(t)}{dt}]_{t=t_{KZ}} \sim 1$$

- This immediately leads to $t_{kz} \sim (\frac{\delta t}{\lambda_0^{1/r}})^{\frac{z\nu}{z\nu+1}}$

- If the system is frozen in the time interval between $-t_{kz}$ and t_{kz} , to lowest order the expectation value of the operator during these times would be the same as that at time t_{kz}

- However the latter is still given by the adiabatic approximation. This in turn has to scale according to its operator dimension

$$< \mathcal{O} > \sim [\xi(-t_{KZ})]^{-\Delta} \sim (\delta t)^{-\frac{\Delta\nu}{z\nu+1}}$$

- An improved version involves scaling functions for correlators

$$< \mathcal{O}(t) >\sim \xi_{kz}^{-\Delta} f(t/t_{kz})$$

$$< \mathcal{O}(\vec{x},t)\mathcal{O}(\vec{x}',t') >\sim \xi_{kz}^{-2\Delta} g(\frac{|\vec{x}-\vec{x}'|}{\xi_{kz}},\frac{|t-t'|}{t_{kz}})$$

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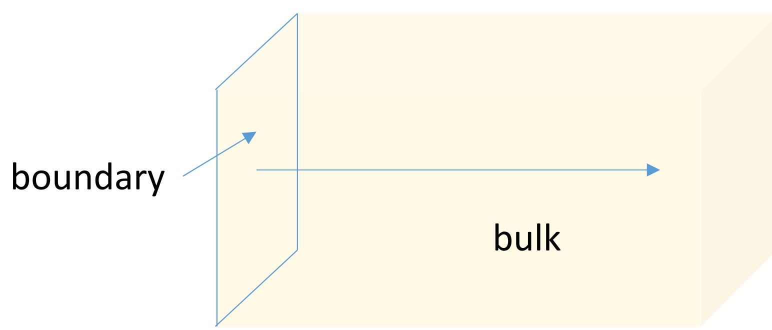
- The initial assumptions of Kibble-Zurek appear too simplistic – nevertheless the scaling results hold for many models.
- Kibble proposed this for thermal transitions in the context of cosmology – Zurek extended this to condensed matter systems - there is some experimental evidence for this. Recently generalized for quantum transitions.
- Experiments in cold atom systems seem to show this kind of scaling.
- However there is little understanding of this in quantum field theory – There is no good understanding of the renormalization group for time dependent systems.

The Holographic Setup

- The AdS/CFT correspondence provides a duality between large N quantum field theories which are deformations of conformal field theories, and theories which contain gravity in asymptotically anti-de Sitter spacetimes.
- The QFT lives on the boundary of AdS space-time. The dual gravitational theory lives in the "bulk" – the interior of AdS
- In many situations, when the QFT is strongly coupled, the dual gravitational description is classical General Relativity with other appropriate bulk fields – a strongly coupled problem in QFT then gets mapped to a classical problem.



- In this regime of strong coupling operators in the field theory are dual to fields in the bulk

scalar	$\mathcal{O}(\vec{x},t) \leftrightarrow \phi(\vec{x},t,z)$	scalar field
vector current	$J_\mu(\vec{x},t) \leftrightarrow A_\mu(\vec{x},t,z)$	gauge field
EM tensor	$T_{\mu\nu}(\vec{x},t,z) \leftrightarrow h_{\mu\nu}(\vec{x},t,z)$	metric perturbation

- Near the boundary one has an asymptotic expansion

$$\phi(\vec{x},t,z) \sim z^{d-\Delta}[\lambda(\vec{x},t) + O(z^2)] + z^{\Delta}[A(\vec{x},t) + O(z^2)]$$

- The integration function $\lambda(\vec{x},t)$ is the coupling for the deformation

$$S = S_{CFT} - \int d^{d-1}x dt \ \lambda(\vec{x},t)\mathcal{O}(\vec{x},t)$$

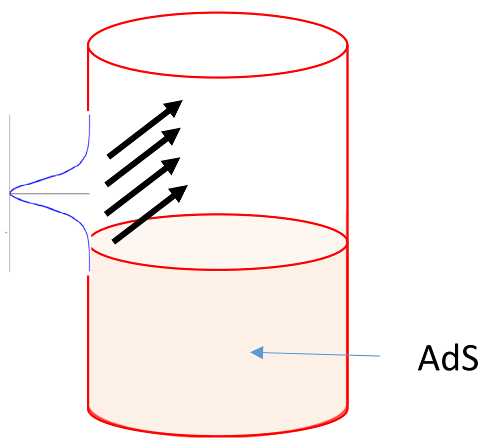
- While the integration function $A(\vec{x},t)$ gives the response

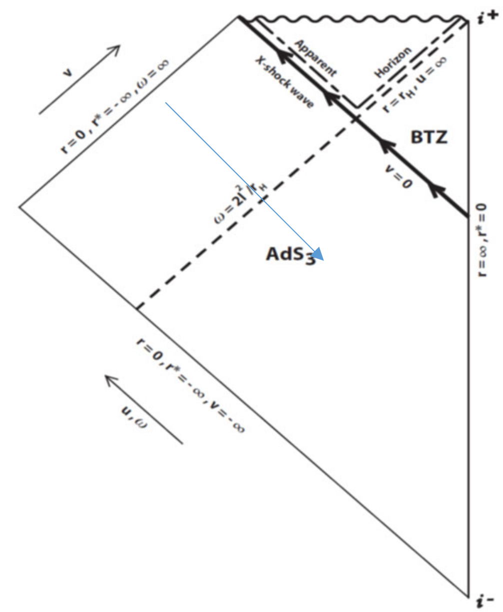
$$< \mathcal{O}(\vec{x},t) >= A(\vec{x},t)$$

- The problem of quantum quench translates into a classical bulk problem with a time dependent boundary condition.
- An important aspect of this relates to thermalization of the field theory.

This becomes dual to formation of black holes in the bulk.

(*Danielsson, Keski-Vakkuri and Kruczenski*; *Chesler and Yaffe*; *Bhattacharyya and Minwalla*; *Balasubramanian et.al*; *Bizon & Rostworoski*; *Buchel, Lehners, Leibling*; *Basu and Krishnan*; *Craps, Evnin, van Hoof*;……)





- Here we will study quantum quench involving a critical point – concentrating on universal scaling behavior

  (1) Identify a bulk theory which is dual to a critical point.

  (2) Impose time dependent boundary conditions on the bulk field which is dual to a relevant operator which crosses or approaches the critical point.

  (3) Calculate the response

Holographic Kibble-Zurek

This has been studied both at zero and non-zero temperature

- *P. Basu and S.R.D., JHEP 1201 (2012) 103*
  P. Basu, D. Das, S.R.D. & T. Nishioka, JHEP 1301 (2013) 107
  P. Basu, D. Das, S.R.D. & K. Sengupta, JHEP 1311 (2013) 186
  S.R.D. & T. Morita, JHEP 1501 (2015) 084.
- *J. Sonner, A. del Campo and W. Zurek, Nat. Comm. 6 (2015) 7405*
- *P. Chesler, A. Garcia-Garcia and H. Liu, Phys. Rev. X (2015) 021015*

- There are many holographic realizations of quantum critical phase transitions which describe superfluidity, magnetism etc.
- In these models, the transition is obtained by tuning some coupling of the boundary field theory, e.g. a chemical potential. In this case the bulk meaning of this chemical potential would be the boundary value of a bulk gauge field.

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- In most cases, the critical point corresponds to the appearance of a zero mode of the corresponding bulk field at the linearized level.
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- When the chemical potential is larger than this critical value, the trivial solution becomes unstable, and the full nonlinear equations have a stable solution which is non-trivial – this is manifested as a nonzero expectation value of the corresponding operator.
- To study quantum quench we make this boundary condition time dependent and solve the bulk equations of motion – from this read off the response in the field theory.
- We want to choose the time dependence such that the coupling crosses a critical point.

- We implemented this in several examples of quench across holographic quantum critical points.
- In all these cases, we showed analytically that in the slow quench regime this zero mode dominates the dynamics in the critical region.
- The zero mode dynamics has a scaling solution – this leads to Kibble-Zurek scaling.

Holographic Fast Quench

- *Buchel, Lehner, Myers and van Niekerk* looked at the opposite regime of a fast but smooth quench, $\delta t \ll m_{gap}^{-1}$, using these holographic setups.
- They found a different kind of scaling behavior for quenches driven by bulk scalars

$$< \mathcal{O} > \sim (\delta t)^{d-2\Delta} \qquad < \mathcal{E} > \sim (\delta t)^{d-2\Delta} \to \infty$$

- These results, first obtained numerically, can be understood by noticing that in this regime the response is dominated by the space-time close to the boundary.

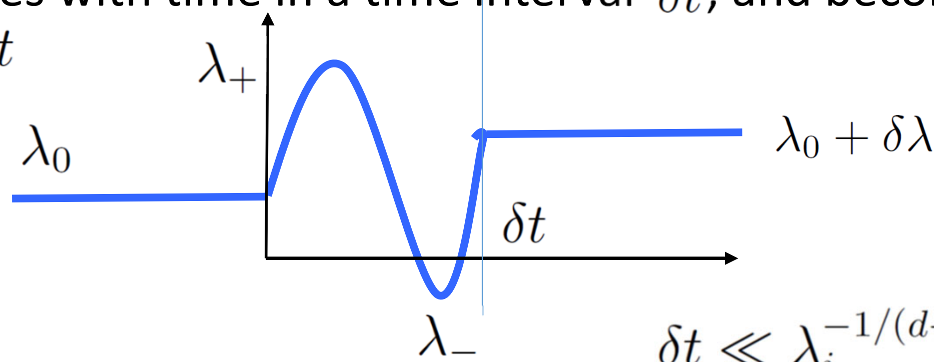
- It turns out that this is in fact a result in any relativistic quantum field theory, regardless of holography.

(S.R.D., Damian Galante and Robert C. Myers)

- Consider a general interacting CFT deformed by some relevant operator with a time dependent coupling

$$S = S_{CFT} - \int dt \int d^{d-1}x \; \lambda(t)\mathcal{O}(\vec{x},t) \qquad \lambda(t) = \lambda_0 F(t/\delta t)$$

- The coupling is at some constant value λ_0 for $t < 0$. At $t = 0$ it smoothly turns on and changes with time in a time interval δt, and becomes a constant quickly after $t = \delta t$



$$\delta t \ll \lambda_i^{-1/(d-\Delta)}, \delta\lambda \qquad i = 0, \pm$$

- Start computing $< \mathcal{O} >$. in perturbation theory. The first few terms are

$$\langle \mathcal{O}(\vec{x},t)\rangle - \langle \mathcal{O}(\vec{x},t)\rangle_{\lambda_0} = -\delta\lambda \int_0^t dt' F\left(t'/\delta t\right) \int d^{d-1}\vec{x}' G_{R,\lambda_0}\left(\vec{x}-\vec{x}', t-t'\right)$$
$$+ \frac{\delta\lambda^2}{2}\int_0^t dt' F\left(t'/\delta t\right)\int d^{d-1}\vec{x}' \int_0^t dt'' F\left(t''/\delta t\right)\int d^{d-1}\vec{x}''\ K_{\lambda_0}\left(t',\vec{x}';t'',\vec{x}'';t,\vec{x}\right)$$

- Where $G_{R,\lambda_0}(\vec{x},t)$ is the retarded Green's function of the initial theory

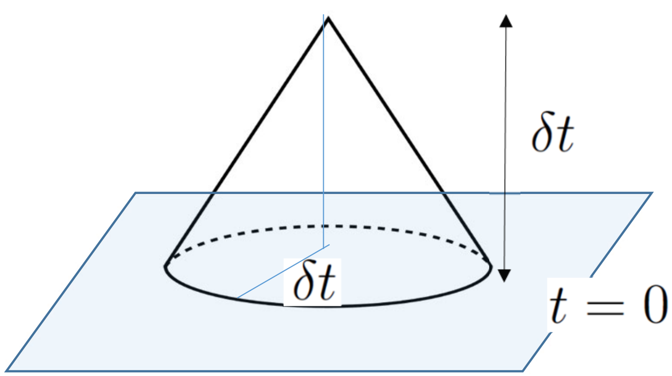
$$G_{R,\lambda_0}(\vec{x},t) = i\theta(t) < 0|[\mathcal{O}(\vec{x},t),\mathcal{O}(0,0)]|0>_{\lambda_0}$$

- $K_{\lambda_0}\left(t',\vec{x}';t'',\vec{x}'';t,\vec{x}\right)$ is a three point function.
- All quantities above are renormalized quantities. This requires counter-terms which depend on $\lambda(t)$ as well as its time derivatives : these counter-terms can be read off, with precise coefficients, from *adiabatic expansions*

- Consider the first term

$$\int_0^t dt' F\left(t'/\delta t\right) \int d^{d-1}\vec{x}' G_{R,\lambda_0}\left(\vec{x}-\vec{x}', t-t'\right)$$

- While the integration over $\vec{x}'$ has been written as over entire space, causality implies that only the region $|\vec{x}-\vec{x}'| \leq t$ has a non-trivial contribution.
- Now suppose we want to calculate the quantity at $t = \delta t$, right at the end of the quench. Then both the space and time intervals which appear in the integral are at most of size δt



- Recall that the scale δt is smaller than all other physical scales in the problem, in particular the scale associated with the deformation of the CFT by λ_0

- Therefore the Green's function $G_{R,\lambda_0}\left(\vec{x}-\vec{x}', t-t'\right)$ is basically the Green's function of the UV conformal field theory.

$$G_{R,\lambda_0}(\vec{x}'-\vec{x}, t'-t) \sim G_{R,CFT}(\vec{x}'-\vec{x}, t'-t) \qquad |\vec{x}'-\vec{x}|, |t'-t| \ll (\lambda_0)^{-1/(d-\Delta)}$$

- This means that in the leading contribution, the only scale which appears in the integral is δt

- This leads to an expression for the response which is of the form

$$\langle \mathcal{O}_\Delta(t)\rangle_{\rm ren} - \langle \mathcal{O}_\Delta(t)\rangle_{{\rm ren},\lambda_0} = (\delta t)^{-\Delta}\left[b_1(t/\delta t)\, g + b_2(t/\delta t)\, g^2 + \cdots\right]$$

- Where we have introduced a dimensionless coupling $g \equiv \delta\lambda\delta t^{d-\Delta}$
- In the fast quench limit this dimensionless coupling is small, so that to leading order we get the universal scaling

$$< \mathcal{O} > \sim (\delta t)^{d-2\Delta}$$

- There is a similar argument for the energy density produced. Since the coupling remains constant after δt this quantity is in fact the net energy produced.

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- There is a similar argument for the energy density produced. Since the coupling remains constant after δt this quantity is in fact the net energy produced.
- Note that the result is completely general and depends only on the properties of the conformal field theory in the UV.
- A more detailed understanding of this general result appears in *Berentsein and Miller* and by *Dymarsky and Smolkin.*
- Explicit calculations in solvable (free) field models.

The Abrupt Limit

- It may appear puzzling that in the $\delta t \to 0$ limit, the fast quench formulae like

$$< \mathcal{O} > \sim (\delta t)^{d-2\Delta}$$

diverge for $2\Delta > d$ .This should be the abrupt quench limit !

- In this limit several results are known in low dimensions – mostly due to *Calabrese and Cardy*. The results are certainly not infinite.
- The reason of course is that we have been considering renormalized quantities, so the quench rate is fast compared to physical mass scales, but slow compared to the UV scale.

$$\Lambda_{UV}^{-1} \ll \delta t \ll m_{gap}^{-1}$$

- To understand this better we have studied solvable models on a lattice.

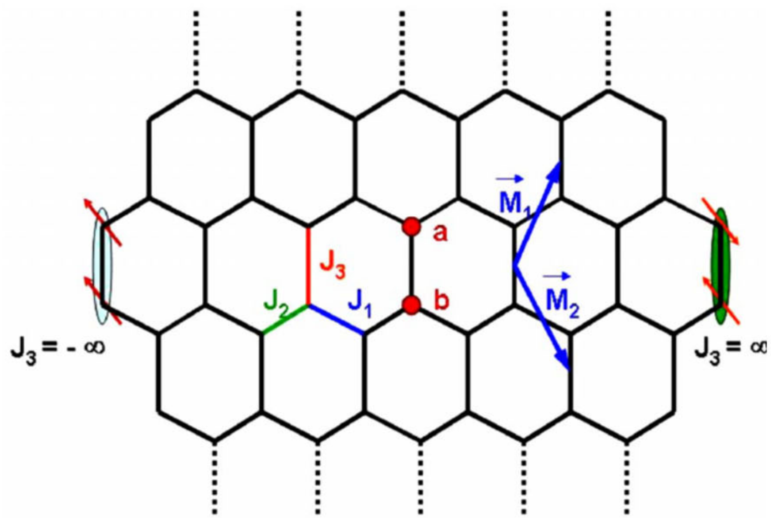
Models on a lattice

(D.Das, S.R.D.,D. Galante, R. Myers and K. Sengupta)

- We have found exactly solvable quench protocols (with couplings which asymptote to constant values) for several solvable lattice models.

$$H = -\sum_n \left[g(t)\sigma^3(n) + \sigma^1(n)\sigma^1(n+1) \right]$$

ISING



$$H_{\text{Kitaev}} = \sum_{j+l=\text{even}} \left[J_1\, \sigma^{(1)}_{j,l}\sigma^{(1)}_{j+1,l} + J_2\, \sigma^{(2)}_{j,l}\sigma^{(2)}_{j-1,l} + J_3\, \sigma^{(3)}_{j,l}\sigma^{(3)}_{j,l+1} \right]$$

KITAEV HONEYCOMB

- Both the models can be written in terms of fermions using Jordan Wigner

$$H = \int \frac{d^D k}{(2\pi)^D}\, \chi^\dagger(\vec{k}) \left[-m(\vec{k},t)\,\sigma_3 + G(\vec{k})\,\sigma_1\right] \chi(\vec{k}) \qquad \chi(\vec{k}) = \begin{pmatrix} \chi_1(\vec{k}) \\ \chi_2(\vec{k}) \end{pmatrix}$$

	d	$m(k,t)$	$G(k)$
Ising	2	$g(t) - \cos k$	$\sin k$
Kitaev	3	$-2(J_3(t) + J_1 \cos k_1 + J_2 \cos k_2)$	$2(J_1 \sin k_1 - J_2 \sin k_2)$

$$\chi_2(\vec{k}) = \chi_1^\dagger(-\vec{k})$$

- The Ising Model has an isolated critical point. The RG flow from this UV fixed point gives a continuum theory of a relativistic Majorana fermion in 1+1 dimensions

$$g = 1 - \epsilon(t)$$

- Introduce a lattice spacing, rescale momenta, couplings

$$p = k/a\,, \qquad m(t) = \epsilon(t)/a\,, \quad \text{and} \quad \psi(p) = a^{1/2}\,\hat{\chi}(a\,p) \qquad \hat{J} = J\,a$$

- Leads to the QFT Hamiltonian

$$H^{\text{cont}}_{\text{Ising}} = 2\hat{J}\int_0^\infty \frac{dp}{2\pi}\,\psi^\dagger(p)\,[m(t)\,\sigma_3 + p\,\sigma_1]\,\psi(p)$$

- Thus a time dependent transverse field of the Ising model is essentially a time dependent mass for a Majorana fermion.

Exactly Solvable Quench Protocol

- Even though the Hamiltonians look quite complicated, it turns out that we can find exact analytic solutions of the operator equations of motion for interesting profiles

$$\begin{pmatrix} g(t) \\ J_3(t) \end{pmatrix} = a + b\tanh(t/\delta t)$$

- Recall the Hamiltonian is of the form

$$H = \int \frac{d^D k}{(2\pi)^D}\, \chi^\dagger(\vec{k}) \left[-m(\vec{k},t)\,\sigma_3 + G(\vec{k})\,\sigma_1\right] \chi(\vec{k})$$

- The positive and negative frequency "in" solutions are

$$U(\vec{k},t) = \begin{pmatrix} -i\partial_t + m(\vec{k},t) \\ -G(\vec{k}) \end{pmatrix} \phi(\vec{k},t) \qquad V(\vec{k},t) = \begin{pmatrix} G(\vec{k}) \\ i\partial_t + m(\vec{k},t) \end{pmatrix} \phi^\star(\vec{k},t)$$

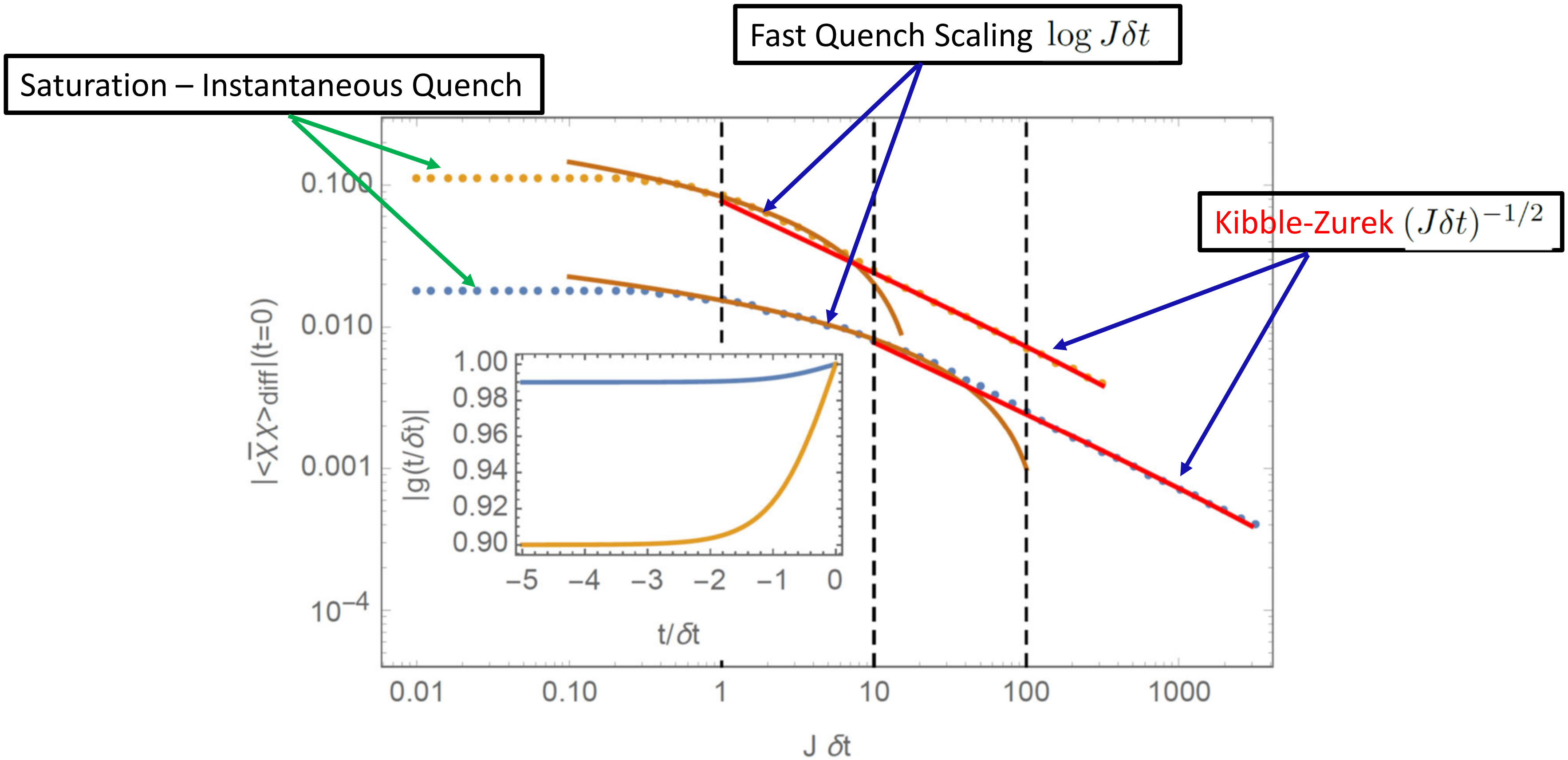
- This is like solving the Dirac equation in momentum space with a "mass" which is both momentum dependent and time dependent

$$m(k,t) = A(\vec{k}) + B\tanh(t/\delta t)$$

	$A(\vec{k})$	B
Ising	$-2J(\cos k - a)$	$2J\,b$
Kitaev	$-2J(\cos k_1 + \cos k_2 - 2\,a)$	$4J\,b$

- The mode functions can be solved exactly for all δt in terms of hypergeometric functions.
- So we can express the response ${}_{in}\langle 0|\bar{\chi}\chi|0\rangle_{in}$ as a momentum integral which we evaluate numerically

Ising Model : Small Amplitude



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- A fast quench regime – as predicted by the continuum quantum field theory – exists in this lattice system when the rate is slow compared to the lattice scale, but fast compared to physical mass scales in the theory – this is something which should be observable in cold atom experiments.

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- For rates slow compared to physical scales one gets, as expected, a Kibble Zurek scaling regime with exponents predicted by the continuum theory

Scaling of non-local quantities

- It turns out that several intersting non-local quantities which characterize quantum systems also show these three regimes of scaling.

  Entanglement Entropy (P. Caputa, S.R.D., M. Nozaki, A. Tomiya)

  Complexity (P. Caputa et.al.; S. Liu)

- It appears that this universality is shared by many quantities.

Non-relativistic theories

- Very often experimental situations involve non-relativistic theories –e.g. atoms in optical traps.
- We would like to learn what happens to these universal scalings in such non-relativistic situations.
- In particular, the argument for fast quench scaling heavily relied on relativistic invariance – one might wonder if there is some version of this in non-relativistic systems as well.
- Many such systems (typically those on a lattice) have an effective light cone provided by the Lieb Robinson bound – so there is a possibility that we may get such scaling regimes as well.
- We will try to work our way up by finding exactly solvable models of quench – hopefully this will teach us some lessons as it did for relativistic theories.

Non-relativistic Fermions in $\pm x^2$ potentials

- We will consider a system of N non-relativistic fermions in 1+1 dimensions with the Hamiltonian

$$H = \int dx\ \psi^\dagger(x)[-\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2} \pm \frac{1}{2}m(t)\nu^2(t)x^2]\psi(x)$$

- The + sign is relevant for harmonic traps. We will see later that the – sign will be relevant for two dimensional strings.
- We have the standard anti-commutators and the particle number constraint

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x - x') \qquad \int_{-\infty}^{\infty} \psi^\dagger(x,t)\psi(x,t) = N$$

- Heisenberg equations of motion is the Schrodinger equation with appropriate initial conditions

$$i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t)$$

- A general solution of the Schrodinger problem can be mapped to a solution of a non-linear equation as follows.
- First introduce a new time variable and a scaled frequency

$$d\tau = \frac{dt}{m(t)} \qquad \omega(\tau) = m(t)\nu(t)$$

- This maps the problem to a constant mass and a time dependent frequency

$$i\frac{\partial\psi(x,\tau)}{\partial\tau} = [-\frac{1}{2}\frac{\partial^2}{\partial x^2} \pm \frac{1}{2}\omega^2(t)x^2]\psi(x,\tau)$$

- This problem has been studied (*Lewis and Risenfeld*; *Pedrosa*; *Ji, Kim and Kim*). Make a change of variables

$$\tau \to T = \int^{\tau} \frac{d\tau'}{\rho(\tau')^2}$$

$$x \to y = \frac{x}{\rho(\tau)}$$

- Define a new field $\Phi(y,T)$ defined by

$$\psi(x,\tau) = \frac{1}{\sqrt{\rho(\tau)}} \exp[-\frac{i}{2}\frac{\partial_\tau \rho(\tau)}{\rho(\tau)}]\ \Phi(y,T)$$

- Obeys the Schrodinger equation

$$i\frac{\partial \Phi(y,T)}{\partial T} = [-\frac{1}{2}\frac{\partial^2}{\partial y^2} \pm \frac{1}{2}y^2]\Phi(y,T)$$

- If the function $\rho(\tau)$ obeys a generalized Ermakov-Pinney equation

$$\partial_\tau^2 \rho(\tau) \pm \omega(\tau)^2 \rho(\tau) = \pm \frac{1}{\rho(\tau)^3}$$

- So the problem of solving the Schrodinger equation with a time dependent mass and a time dependent frequency can be reduced to the equation with constant mass and frequency – and a solution of this nonlinear equation.

- A general solution of the Ermakov-Pinney equation is of the form

$$\rho(\tau)^2 = Af(\tau)^2 + 2Bf(\tau)g(\tau) + Cg(\tau)^2$$

- Where $f(\tau), g(\tau)$ are two linearly independent solutions of

$$\partial_\tau^2 X \pm \omega(\tau)^2 X = 0$$

- And the constants are related by

$$AC - B^2 = \pm \frac{1}{Wr(f,g)^2}$$

- We have therefore reduced the solution of the problem to solutions of the classical equations of motion for the same Hamiltonian.

- Using the solution of the field equation one can now quantize the theory canonically

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n \, \psi_n(x,t)$$

$$\{a_m, a_n^\dagger\} = \delta_{mn} \quad \{a_n, a_m\} = \{a_m^\dagger, a_n^\dagger\} = 0$$

- For quench protocols which asymptote to constant frequencies at early and late times the "in" modes have the property

$$\text{Lim}_{t\to\infty}\psi_n(x,t) \sim e^{-i\alpha t} \qquad \alpha > 0$$

- While the function $\rho(\tau)$ must have the initial condition

$$\text{Lim}_{\tau\to-\infty}^{\rho(\tau)} = \rho_{in} = \frac{1}{\sqrt{m_{in}\nu_{in}}} = \frac{1}{\sqrt{\omega_{in}}}$$

- This defines the Heisenberg picture "in" vacuum state

$$a_n|in\rangle = 0 \qquad n \geq N$$

$$a_n^\dagger|in\rangle = 0 \qquad 0 \leq n \leq N-1$$

Observables

- In fact many interesting observables can be expressed entirely in terms of the function $\rho(\tau)$
- Consider for example the expectation value of the quenched operator

$$\mathcal{O} = \int_{\infty}^{\infty} x^2 \psi^\dagger(x,\tau)\psi(x,\tau)$$

- In the “in” state this is

$$\langle 0|\mathcal{O}(\tau)|0\rangle = \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} dx\ x^2\ \psi_n^\star(x,\tau)\psi_n(x,\tau)$$

$$= \rho^2(\tau) \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} dY Y^2 \phi_n^\star(Y)\phi_n(Y) = \rho(\tau)^2 \sum_{n=0}^{N-1} (n+1/2) = \boxed{\frac{N^2}{2}\rho(\tau)^2}$$

Entanglement Entropy

- Another interesting quantity is the entanglement entropy of some region A.
- For free fermions it turns out that this quantity can be expressed in terms of cumulants of the particle number distribution. (*Kilch and Levitov*)
- For large N the leading contribution is the dispersion

$$S_A(\tau) = \frac{\pi^2}{3}[\langle N_A(\tau)^2\rangle - \langle N_A(\tau)\rangle^2]$$

$$N_A(\tau) = \int_A dx \psi^\dagger(x,\tau)\psi(x,\tau)$$

- In the "in" state this can be further simplified *(Collura, Sotiriadis & Calabrese)*

$$S_A(\tau) = \langle in|N_A(\tau)|in\rangle - \int_A dx \int_A dy |C(x,y,\tau)|^2$$

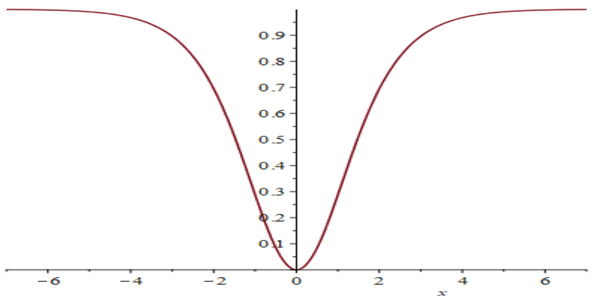
$$C(x,y,\tau) = \langle in|\psi^\dagger(x,\tau)\psi(y,\tau)|in\rangle$$

- Using the mapping described above this can again be expressed in terms of the results for fermions in a potential with constant frequency, but with an interval rescaled by the function $\rho(\tau)$

$$S_A[\omega(\tau)] = S_{A_P}[\omega = 1] \qquad A_P : \frac{a}{\rho(\tau)} \leq x \leq \frac{b}{\rho(\tau)}$$

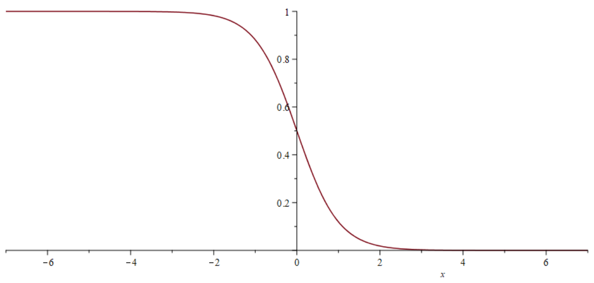
- We have investigated explicit solutions for two interesting protocols
- The first is a "Cis-Critical Protocol (CCP)"

$$\omega(\tau)^2 = \omega_0^2 \ \tanh^2(t/\delta t)$$



- The second is a End-Critical protocol (ECP)

$$\omega^2(\tau) = \frac{1}{2}\omega_0^2 \left(1 - \tanh\frac{\tau}{\delta t}\right)$$



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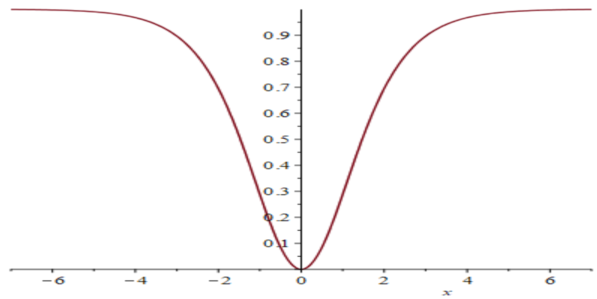

- The "in" classical solution is

$$f_{CCP}(\tau) = \frac{1}{\sqrt{2\omega_0}} \frac{2^{i\omega_0\delta t}\cosh^{2\alpha}(\tau/\delta t)}{E_{1/2}\tilde{E}'_{3/2} - E'_{1/2}\tilde{E}_{3/2}} \times$$

$$\left\{ \tilde{E}'_{3/2} \; {}_2F_1(a, b; \frac{1}{2}; -\sinh^2\frac{\tau}{\delta t}) + E'_{1/2}\sinh\frac{\tau}{\delta t} \; {}_2F_1(a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; -\sinh^2\frac{\tau}{\delta t}) \right\}$$

- where

$$\alpha = \frac{1}{4}[1 + \sqrt{1 - 4\omega_0^2\delta t^2}]$$

$$a = \alpha - \frac{i}{2}\omega_0\delta t, b = \alpha + \frac{i}{2}\omega_0\delta t$$

$$E_{1/2} = \frac{\Gamma(1/2)\Gamma(b-a)}{\Gamma(b)\Gamma(1/2-a)}, \tilde{E}_{3/2} = \frac{\Gamma(3/2)\Gamma(b-a)}{\Gamma(b+1/2)\Gamma(1-a)}$$

$$E'_c = E_c(a \leftrightarrow b)$$

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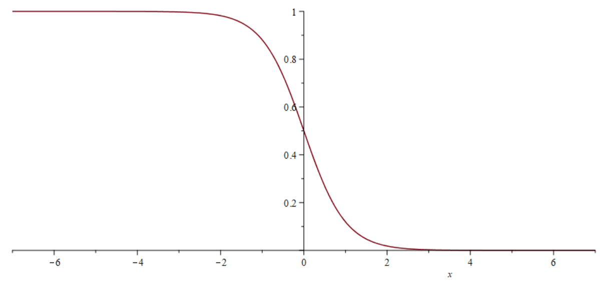
$$\omega^2(\tau) = \frac{1}{2}\omega_0^2\left(1 - \tanh\frac{\tau}{\delta t}\right)$$



- The "in" classical solution is

$$f_{ECP} = \frac{1}{\sqrt{2\omega_{in}}}\exp[-i\omega_+\tau - i\omega_-\delta t\log(2\cosh(\tau/\delta t))]$$

$$_2F_1[1 + i\omega_-\delta t, i\omega_-\delta t; 1 - i\omega_{in}\delta t; \frac{1}{2}(1 + \tanh(\tau/\delta t))]$$

$$\omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in})$$

Scaling for CCP

- At times in the middle of the quench or just when the quench ends we find different scaling regimes.
- We find answers consistent with Kibble Zurek scaling for $\omega_0 \delta t \gg 1$

$$\langle O(0) \rangle \sim \frac{\sqrt{\pi}}{2} N^2 \sqrt{\frac{\delta t}{\omega_0}}$$

- In the fast regime $\omega_0 t \ll \omega_0 \delta t \ll 1$

$$\langle \mathcal{O}(t) \rangle \approx \frac{N^2}{2\omega_0} \left\{ 1 + 2\log 2 \cdot \omega_0^2 \delta t^2 + 2\omega_0^2 \delta t \cdot t + \mathcal{O}(\omega_0^4 \delta t^4, \frac{t^2}{\delta t^2}) \right\}$$

- This is consistent with perturbation theory. However unlike relativistic theories the linear response term is not proportional to ω_0^2

Early time response : ECP

- For end critical protocols it is more appropriate to look at response for a range of times. Now at any given time a sufficiently slow quench will always be adiabatic – since the gapless point is approached asymptotically.
- For slightly higher quench rates there is a Kibble-Zurek regime where

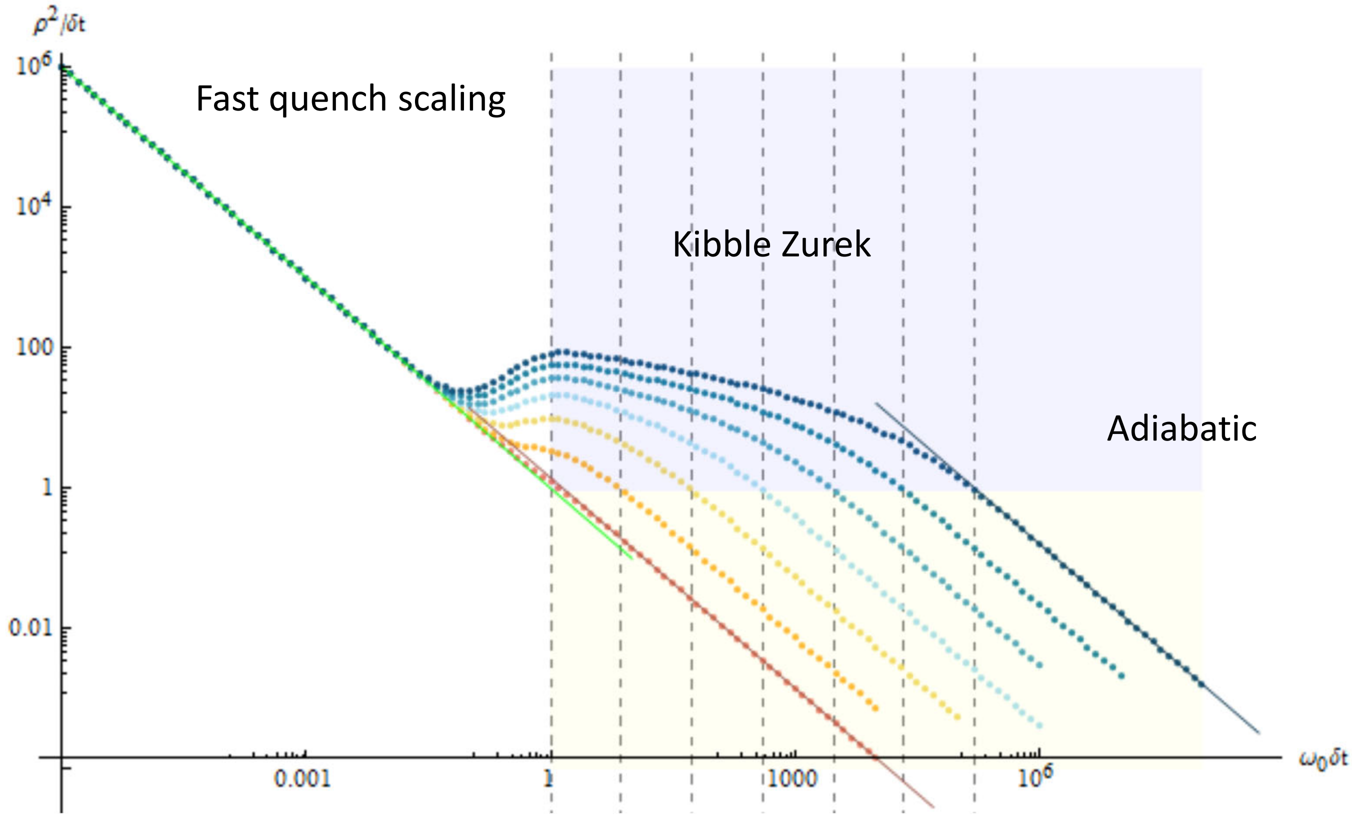
$$< \mathcal{O} > \sim N^2 \rho(t_{KZ})^2 = \frac{N^2}{\omega_{KZ}} \sim N^2 \delta t$$

- An expansion of the exact answer actually reveals a log correction

$$\rho^2(\tau) \sim \delta t \left[\frac{2}{\pi} \left(- \log \omega_0 \delta t + \log 2 - \gamma_E + \frac{\tau}{\delta t} \right)^2 + \frac{\pi}{2} \right] \sim \mathcal{O}(1)$$

- In the fast quench regime $\omega_0 \delta t \ll \omega_0 \tau \ll 1.$ we get

$$\rho^2(\tau) \sim \omega_0 \delta t^2 \left(-\frac{\zeta(3)}{4} \omega_0^2 \delta t^2 + \frac{\tau}{\delta t} \right)^2 + \frac{1}{\omega_0} = \frac{1}{\omega_0} + \omega_0 \tau^2 - \frac{\zeta(3)}{2} (\omega_0 \delta t)^3 \tau.$$



$\rho^2/\delta t$
10^6
10^4
100
1
0.01
Fast quench scaling
Kibble Zurek
Adiabatic
0.001
1
1000
10^6
$\omega_0\delta t$

Connection to Two Dimensional Strings

- Fermions in an inverted harmonic oscillator potential is well known to provide a non-perturbative description of 2 dimensional string theory – via its connection to Matrix Quantum Mechanics.
- In the double scaled limit the Hamiltonian is given by

$$H = \frac{1}{2}\int dx\ \psi^\dagger[-\frac{1}{g_s}\partial_x^2 - g_s x^2]\psi$$

- Where g_s is the scaled string coupling which is held fixed in the large N double scaled limit. (The physical string coupling is position dependent, but proportional to g_s
- A time-dependent coupling is then the problem with a time dependent mass and a constant frequency.
- In this case we get interesting examples of two dimensional cosmologies.

Epilouge

- Our results show that there is interesting universal behavior for quench rates at intermediate scales between the UV and physical mass scales.
- This result arose from an interesting interplay of holography and standard field theoretic techniques.
- We have mostly concentrated on the behavior near the *end-point* of the quench. While this gives late time results for quantities like the energy density- it would be interesting to explore how this universality manifests in other quantities at late times, and their implications to thermalization.
- In particular, it would be interesting to see how this appears in the time evolution of entanglement which has been so far studied for abrupt changes.

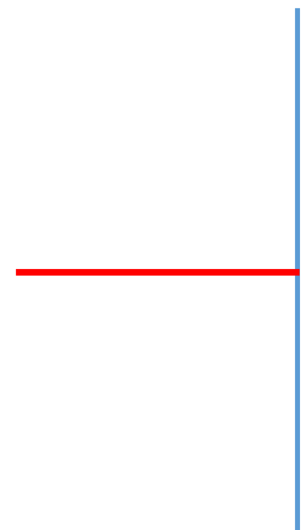
Holographic Cosmologies

- In the holographic context, there are situations where a quantum quench on the boundary does not lead to black hole formation.
- Rather, the bulk space-time develops a space-like region of high curvature akin to cosmological singularities. (*A. Awad, S.R.D., K. Narayan, S. Trivedi : 2008,2009*).
- One example is the AdS-Kasner

$$ds_b^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + \sum_{a=1}^{(d-1)} t^{2p_a} (dx^a)^2,$$

$$\Phi(t) = \alpha \log |t|$$

$$\sum_{a=1}^{d-1} p_a = 1, \qquad \sum_{a=1}^{d-1} p_a^2 = 1 - \frac{\alpha^2}{2}$$



- A time dependent dilaton means that the coupling of the boundary theory is time dependent – so a quantum quench. There are also solutions where the dilaton is bounded

$$ds^2 = \frac{1}{z^2}dz^2 + \left[|\sinh(2t)|\left(-dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_2^2\right)\right]$$

$$\Phi = -2\sqrt{3}\tanh^{-1}(e^t).$$

- These backgrounds were studied in the past with the hope that the boundary theory provides a perfectly well-defined time evolution, thus "resolving" the singularity.
- There was no definitive conclusion about this.
- In recent years this problem has been revisited in several contexts. The aim has been to find what are the signatures of such a bulk singularity in the boundary theory.
- *Englehardt & Horowitz (2014,2015,2016); Barbon & Rabinovici (2015,2018); Frenkel, Hartnoll, Kruthoff & Shi (2020); Hartnoll, Horowitz, Kruthoff & Santos (2021).*

- One interesting question concerns the behavior of complexity of the quantum state near the singularity.
- We recently investigated this issue using a particular notion of complexity.
- The main result is that the complexity decreases as one approaches the singularity in a universal manner – independent of the Kasner exponents. (P. Caputa, D. Das and S.R.D. (2022)).
- This is consistent with the fact that the dimensionality of the Hilbert space decreases as one approaches the singularity.

Thank You