6 December 2022, British University in Egypt, Cairo

Global Portraits of Inflation

Laur Järv
University of Tartu, Estonia

with Alexey Toporensky (Moscow)

2104.10183, Eur.Phys.J.C 82 (2022) 2, 179

with Joosep Lember (Tartu)

2104.14258, Universe 7 (2021) 6, 179

with Alexey Golovnev, ... (Cairo)

work in progress

European Union
European Regional
Development Fund
Investing
in your future

University of Tartu, Estonia

Estonia

- 1.4 million people
- 45,000 km^2
- language Estonian

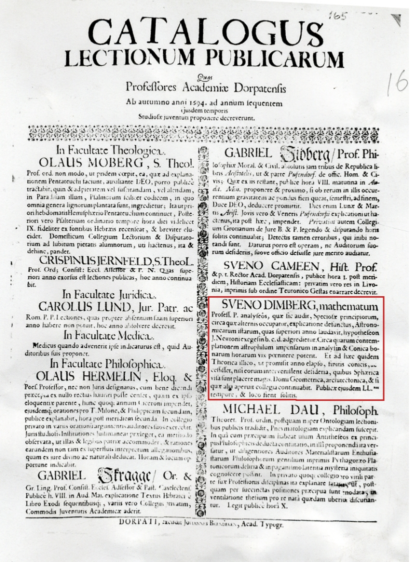
University of Tartu

- Founded 1632
- 13,000 students
- 1,700 academic staff
- Faculties
  - Arts and Humanities
  - Medicine
  - Science and Technology
  - Social Sciences

Sven Dimberg and teaching Newton's theory

Sven Dimberg (1661-1731)

- Born in Sweden, father was a priest, who had studied in Tartu;
- studied in Turku and Uppsala universities;
- travelled in Europe (England);
- University of Tartu professor of mathematics (1690-1699),
- ordered a telescope from England for the "pride of the university",
- gave lectures on Newton's *Principia* in 1693/94-1694/95 and 1696/97-1697/98, first in the world!



CATALOGUS
LECTIONUM PUBLICARUM
Quas
Professores Academiæ Dorpatensis
Ab autumno anni 1694. ad annum sequentem
ejusdem temporis
Studiosæ juventuti proponere decreverunt.

In Facultate Theologica.
OLAUS MOBERG, S. Theol.

CRISPINUS JERNFELD, S.Theol.

In Facultate Juridica.
CAROLUS LUND, Jur. Patr. ac Rom. P. P.

In Facultate Medica.

In Facultate Philosophica.
OLAUS HERMELIN, Eloq. & Poes. Professor

GABRIEL Skragge / Or. &

GABRIEL Sjöberg / Prof. Philosophiæ Moral. & Civil.

SVENO CAMEEN, Hist. Prof. & p. t. Rector Acad. Dorpatensis, publice hora 1. post meridiem, Historiam Ecclesiasticam: privatim vero res in Livonia, inprimis sub ordine Teutonico Gestas enarrare decrevit.

SVENO DIMBERG, mathematum Profess. P. analyseos, quæ sic audit, Speciosæ principiorum, circa quæ alternis occupatur, explicatione defunctus, Astronomicarum istarum, quas superiori anno laudavit, hypotheseon J. Newtoni exegesin b. c. d. adgredietur. Circa quarum contemplationem astrophilum impensarum in analysin & Conica bonarum horarum vix pœnitere poterit. Et ad hæc quidem Theorica illico, ut promisit anno elapso, finitis conicis, accessisset, nisi eorum intervenissent desideria, quibus Sphærica visa sunt placere magis. Domi Geometrica, architectonica, & si quæ alia aperuit collegia continuabit. Publicæ ejusdem LL. tempore, & loco fient solitis.

MICHAEL DAU, Philosoph. Theoret. Prof. ordin. postquam nuper Ontologiam lectionibus publicis tradidit, Pnevmatologiam explicandam suscepit.

DORPATI, excud. Johannes Brendeken, Acad. Typogr.

Sven Dimberg and teaching Newton's theory

SVENO DIMBERG, mathematum Profess. P. analyseôs, quæ sic audit, Speciosæ principiorum, circa quæ alternis occupatur, explicatione defunctus, Astronomicarum istarum, quas superiori anno laudavit, hypotheseon J. Newtoni exegesin b. c. d. adgredietur. Circa quarum contemplationem astrophilum impensarum in analysin & Conica bonarum horarum vix pœnitere poterit. Et ad hæc quidem Theorica illico, ut promisit anno elapso, finitis conicis, accessisset, nisi eorum intervenissent desideria, quibus Sphærica visa sunt placere magis. Domi Geometrica, architectonica, & si quæ alia aperuit collegia continuabit. Publicæ ejusdem LL. ** tempore, & loco fient solitis.

Lumiste, Ü., Piirimäe, H. (2001). Newton's *Principia* in the Curricula of the University of Tartu (Dorpat) in the Early 1690s. In: Vihalemm, R. (eds) Estonian Studies in the History and Philosophy of Science. Boston Studies in the Philosophy of Science, vol 219. Springer, Dordrecht.

F. G. W. von Struve and distance to the stars

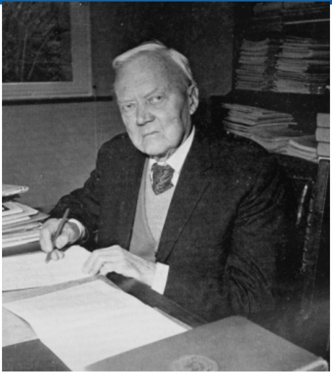
Friedrich Georg Wilhelm von Struve (1793-1864)

- born in Germany, came to his elder brother in Tartu to escape military recruitment,
- studied philology and astronomy in University of Tartu (PhD 1813),
- University of Tartu professor of mathematics and astronomy 1820-1939,
- ordered from Fraunhofer the most powerful telescope of that time,
- measured distance to a star (α Lyrae), and was the first to publish a reasonably accurate result:
  - *Stellarum duplicium et multiplicium mensurae micrometricae per magnum Fraunhoferi tubum annis a 1824 ad 1837 in specula Dorpatensi institutae*, Petropoli Ex Typographia Academica 1837.

Ernst Öpik and distance to the galaxies

Ernst Julius Öpik (1893–1985)

- born in Estonia, studied astronomy in Moscow, PhD from Tartu;
- Head of the Tartu Observatory, visited Harvard every winter 1930-34;
- 1944 left to the West, since 1948 at Armagh Observatory (Northern Ireland) and since 1956 also at the University of Maryland.
- estimated the distance to a galaxy (more accurately than Hubble)
    - "An Estimate of the Distance to the Andromeda Nebula" *The Astrophysical Journal* 1922, 55, pp 406-410

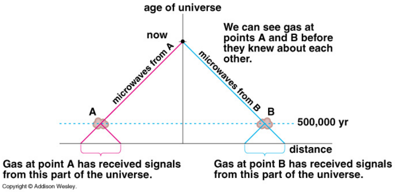


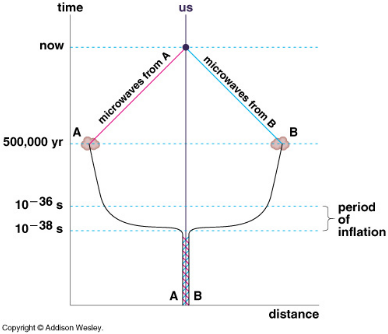
Jaan Einasto and dark matter

Jaan Einasto (born 1929)

- Born in Tartu, studied at University of Tartu, works at Tartu Observatory
- Dark matter in galaxies
  J Einasto, A Kaasik, E Saar, *Dynamic evidence on massive coronas of galaxies*, Nature 250 (5464), 309-310 (1974)
- Large scale structure
  IB Zeldovich, J Einasto, SF Shandarin *Giant voids in the universe*, Nature 300, 407-413 (1982)

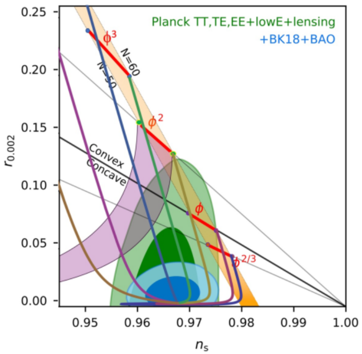
Early Universe inflation

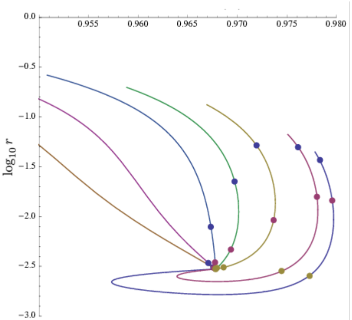




- To solve the horizon problem, expect a period of significant accelerated expansion at early Universe.
- Then consider perturbations.

Inflation, the best evidence for extended gravity?





Kallosh, Linde 2110.10902, data from BICEP, Keck 2110.00483

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ (1 + \xi\phi^2)R - \partial_\mu\phi\,\partial^\mu\phi - 2V(\phi) \right\} , \qquad (1)$$

- orange band: $\xi = 0$, monomial potential $V(\phi) \sim \phi^n$ ruled out,
- purple band: hilltop inflation $\xi = 0$, $V(\phi) \sim 1 - \phi^4$ ruled out,
- lines: nonminimal coupling $\xi \neq 0$, $V(\phi) \sim \phi^n$, attractor $\xi \to \infty$.

Overview

Inflation

- must be an attractor, but not a fixed point,
- rather realized by a heteroclinic orbit in phase space from a saddle or nonhyperbolic point to an attractor point
  c.f. Alho, Uggla 1406.0438, ...

In this work we introduce a physically motivated set of variables and

- draw global phase portraits in the Jordan frame which
- clearly distinguish different asymptotic regimes,
- reveal the inflationary attractor orbit,
- comment on the slow roll approximation in the Jordan frame,
- visualize the range of good initial conditions (> 50 efolds).

Can apply to other models, e.g.

- How to construct the analogue of Higgs inflation in teleparallel gravity?

First paper (LJ, A. Toporensky 2104.10183)

- A scalar field nonminimally coupled to curvature (Jordan frame),

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ F(\phi) R - \partial_\mu \phi \, \partial^\mu \phi - 2V(\phi) \right\} , \qquad (2)$$

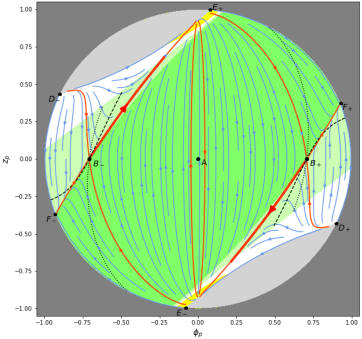
- in flat FLRW cosmology,
- and use dynamical variables (and evolution in e-folds $N = \ln a$)
  Dutta, LJ, Khyllep, Tõkke 2007.06601

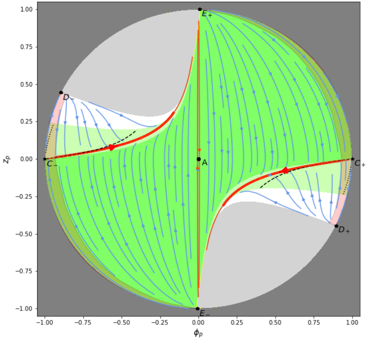
$$\phi , \qquad z = \frac{\dot{\phi}}{H} = \frac{d\phi}{dN} \qquad (3)$$

- study global portraits in Poincaré compactification

$$\phi_p = \frac{\phi}{\sqrt{1 + \phi^2 + z^2}} , \qquad z_p = \frac{z}{\sqrt{1 + \phi^2 + z^2}} . \qquad (4)$$

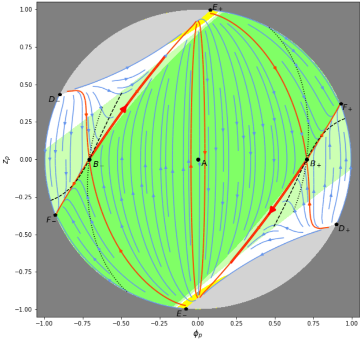
For example: nonminimal coupling $F = 1 + \xi\phi^2$,
quadratic $V = \frac{m^2}{2}\phi^2$ (left), quartic $V = \frac{\lambda}{4}\phi^4$ (right) potential.

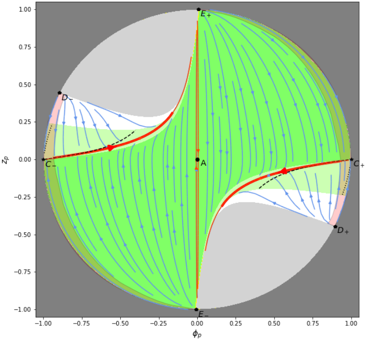




Green – superacceleration, light green – acceleration, white – deceleleration, yellow – superstiff expansion, grey – unphysical region.

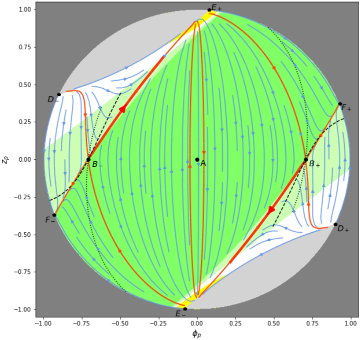
Why is it interesting : asymptotic regimes

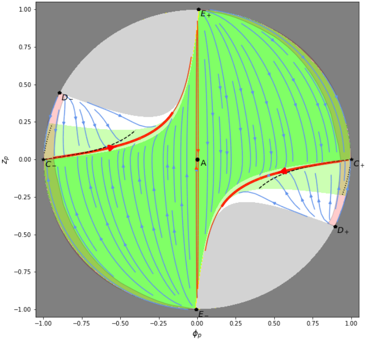




1. Variables (ϕ, z) distinguish different asymptotic regimes: fixed points C, F - asymptotic de Sitter, D, E - kinetic dominated, otherwise mapped to the same point in $(\phi, \dot{\phi})$
   c.f. Carloni, Capozziello, Leach, Dunsby gr-qc/0701009; Sami, Shahalam, Skugoreva, Toporensky, 1207.6691; Skugoreva, Toporensky, Vernov 1404.6226.

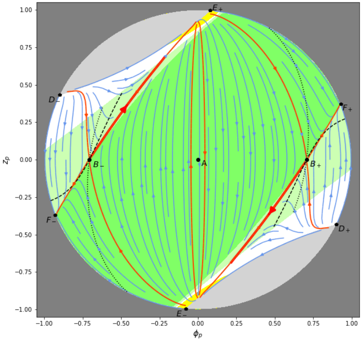
Why is it interesting : inflationary attractor trajectory

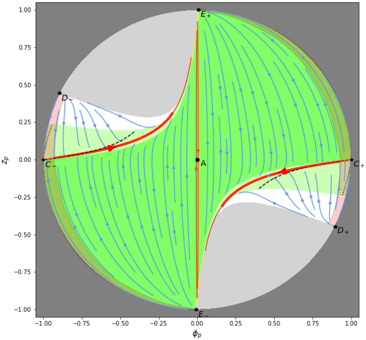




2. Clearly identify inflation as a heteroclinic orbit $B \to A$ or $C \to A$, not realized before in easy to interpret variables, cf. Alho, Uggla 1406.0438, ...

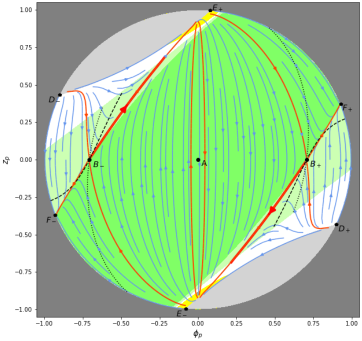
Why is it interesting : slow roll approximation

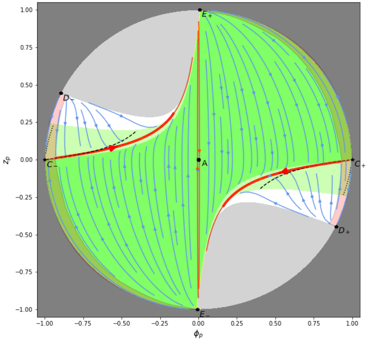




3. Leading inflationary orbit is approximated
   - − not so well by "generalised slow roll" in the Jordan frame (dotted),
   - + but much better by our mechanical analogue [slide 23] (dashed line) which matches Einstein frame slow roll translated into Jordan frame

     c.f. Akın, Arapoglu, Yükselci 2007.10850, Karčiauskas, Terente Díaz 2206.08677





4. Can see the range of "good" initial conditions (trajectories) that lead to at least 50 e-folds of accelerated expansion (pink shadow).
   - For increasing ξ the range shrinks for quadratic, but enlarges for quartic potentials.

Second paper: teleparallel inflation (2104.14258)

- Take a scalar field nonminimally coupled to torsion in the teleparallel framework Geng, Lee, Saridakis, Wu 1109.1092 ; Hohmann, LJ, Ualikhanova 1801.05786

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ F(\phi) T + \partial_\mu \phi \, \partial^\mu \phi - 2V(\phi) \right\} \tag{5}$$

and do the same analysis.

- Incidentally, the background flat FLRW cosmology is the same for a scalar field nonminimally coupled to nonmetricity in the symmetric teleparallel framework

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ -F(\phi) Q + \partial_\mu \phi \, \partial^\mu \phi - 2V(\phi) \right\} . \tag{6}$$

LJ, Rünkla, Saal, Vilson 1802.00492

Aside: Metric-affine geometries

LJ, Rünkla, Saal, Vilson 1802.00492

Decomposition of connection

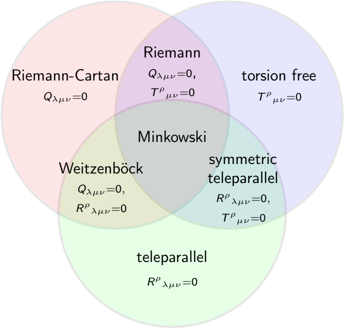
$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {}^{\lambda}{}_{\mu\nu} \right\} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \,,$$

where

$$\begin{aligned}
\left\{ {}^{\lambda}{}_{\mu\nu} \right\} &\equiv \frac{1}{2} g^{\lambda\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right) \,, \\
K^{\lambda}{}_{\mu\nu} &\equiv \frac{1}{2} g^{\lambda\beta} \left(T_{\mu\beta\nu} + T_{\nu\beta\mu} + T_{\beta\mu\nu} \right) \,, \\
L^{\lambda}{}_{\mu\nu} &\equiv \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu} \right)
\end{aligned}$$

and

$$\begin{aligned}
\text{torsion } T^{\lambda}{}_{\mu\nu} &\equiv \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \\
\text{nonmetricity } Q_{\rho\mu\nu} &\equiv \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\beta}{}_{\rho\mu} g_{\beta\nu} - \Gamma^{\beta}{}_{\rho\nu} g_{\mu\beta} \,, \\
\text{curvature } R^{\sigma}{}_{\rho\mu\nu} &\equiv \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\sigma}{}_{\mu\alpha} - \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\sigma}{}_{\nu\alpha} \,.
\end{aligned}$$



Different assumptions about the connection restrict the generic metric-affine geometry.

Three equivalent theories

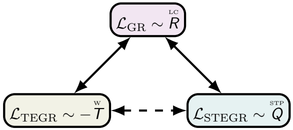
Can express curvature scalar

LJ, Rünkla, Saal, Vilson 1802.00492

$$R \;=\; \overset{\text{LC}}{R} + M^{\alpha}{}_{\nu\rho} M^{\mu}{}_{\mu\alpha} g^{\nu\rho} - M^{\alpha}{}_{\mu\rho} M^{\mu}{}_{\nu\alpha} g^{\nu\rho} + \overset{\text{LC}}{\nabla}_{\mu} \left(M^{\mu}{}_{\nu\rho} g^{\nu\rho} - M^{\nu}{}_{\nu\rho} g^{\mu\rho} \right)$$

where

$$M^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \,.$$



Three equivalent theories

	Connection	Action
General relativity	no torsion, no nonmetricity	$R = \overset{\text{LC}}{R}$
Teleparallel gravity	no curvature, no nonmetricity	$\overset{\text{LC}}{R} = -\overset{\text{W}}{T} - 2\overset{\text{LC}}{\nabla}_{\alpha} \overset{\text{W}}{T}{}^{\alpha}$
Symmetric teleparallel grav.	no curvature, no torsion	$\overset{\text{LC}}{R} = \overset{\text{STP}}{Q} - \overset{\text{LC}}{\nabla}_{\alpha} (\overset{\text{STP}}{Q}{}^{\alpha} - \overset{\text{STP}}{\tilde{Q}}{}^{\alpha})$

where

$$T = \frac{1}{4} T_{\alpha\beta\gamma} T^{\alpha\beta\gamma} + \frac{1}{2} T_{\alpha\beta\gamma} T^{\gamma\beta\alpha} - T_{\alpha} T^{\alpha} \,, \qquad T_{\mu} \equiv T^{\alpha}{}_{\mu\alpha} = -T^{\alpha}{}_{\alpha\mu} \,,$$

$$Q = -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma\beta\alpha} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \tilde{Q}^{\alpha} \,, \quad Q_{\mu} \equiv Q_{\mu}{}^{\alpha}{}_{\alpha} \,, \quad \tilde{Q}^{\mu} \equiv Q_{\alpha}{}^{\alpha\mu} \,.$$

Second paper: teleparallel inflation (2104.14258)

- Take a scalar field nonminimally coupled to torsion in the teleparallel framework Geng, Lee, Saridakis, Wu 1109.1092 ; Hohmann, LJ, Ualikhanova 1801.05786

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,\{F(\phi)T + \partial_\mu\phi\,\partial^\mu\phi - 2V(\phi)\} \tag{7}$$

  and do the same analysis.

- Incidentally, the background flat FLRW cosmology is the same for a scalar field nonminimally coupled to nonmetricity in the symmetric teleparallel framework LJ, Rünkla, Saal, Vilson 1802.00492

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,\{-F(\phi)Q + \partial_\mu\phi\,\partial^\mu\phi - 2V(\phi)\}\ . \tag{8}$$

Couple the scalar field nonminimally to "dynamical" part of the gravity action, not to the boundary term.

Flat FLRW cosmological equations

Scalar-curvature (scalar-tensor) gravity

$$\begin{aligned} 3FH^2 &= \frac{\dot{\phi}^2}{2} + V - 3F_{,\phi}H\dot{\phi} && (9) \\ -2F\dot{H} &= \dot{\phi}^2 + F_{,\phi\phi}\dot{\phi}^2 - F_{,\phi}H\dot{\phi} + F_{,\phi}\ddot{\phi} && (10) \\ \ddot{\phi} + 3H\dot{\phi} &= -V_{,\phi} + 3F_{,\phi}(2H^2 + \dot{H}) && (11) \end{aligned}$$

Scalar-torsion and scalar-nonmetricity gravity (after satisfying the connection equations)

$$\begin{aligned} 3FH^2 &= \frac{\dot{\phi}^2}{2} + V && (12) \\ -2F\dot{H} &= \dot{\phi}^2 + 2F_{,\phi}H\dot{\phi} && (13) \\ \ddot{\phi} + 3H\dot{\phi} &= -V_{,\phi} - 3F_{,\phi}H^2 && (14) \end{aligned}$$

- minimally coupled limit $F = 1$ the same
- otherwise different theories

Generic method to find fixed points in the Jordan frame

- Substitute in $\dot{H}$ to write the scalar field equation as

$$\ddot{\phi} = \frac{1}{m_{\mathrm{eff}}}(-V_{\mathrm{eff},\phi}) - 3\dot{\phi}H - \dot{\phi}^2(\text{extra friction})\,. \qquad (15)$$

- Fixed points

	Effective mass m_{eff}	Effective potential V_{eff}	Fixed point condition	Stability condition condition
Minimally coupled	1	V	$\frac{V_{,\phi}}{V} = 0$	$\frac{V_{,\phi\phi}}{V} > 0$
Scalar-curvature	$\frac{2F+3F_{,\phi}^2}{F^3}$	$\frac{V}{F^2}$	$\frac{1}{m_{\mathrm{eff}}}\frac{V_{\mathrm{eff},\phi}}{V} = 0$	$\frac{1}{m_{\mathrm{eff}}}\frac{V_{\mathrm{eff},\phi\phi}}{V} > 0$
Scalar-torsion	F	FV	$\frac{1}{m_{\mathrm{eff}}}\frac{V_{\mathrm{eff},\phi}}{V} = 0$	$\frac{1}{m_{\mathrm{eff}}}\frac{V_{\mathrm{eff},\phi\phi}}{V} > 0$

Previous discussions on the effective potential Chiba, Yamaguchi 0807.4965; Skugoreva, Toporensky, Vernov 1404.6226; Skugoreva,Toporensky 1605.01989;
LJ et al. 1612.06863; Dutta, LJ, Khyllep, Tõkke 2007.06601

Generic method to find slow roll in the Jordan frame

- Substitute in $\dot{H}$ to write the scalar field equation as

$$\ddot{\phi} = \frac{1}{m_{\text{eff}}}(-V_{\text{eff},\phi}) - 3\dot{\phi}H - \dot{\phi}^2(\text{extra friction})\,. \tag{16}$$

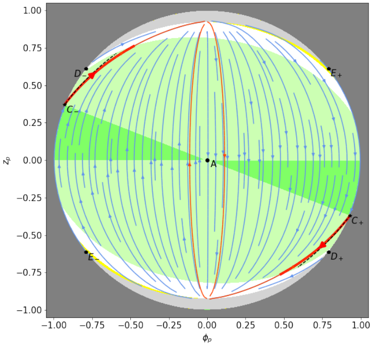
- Slow roll condition

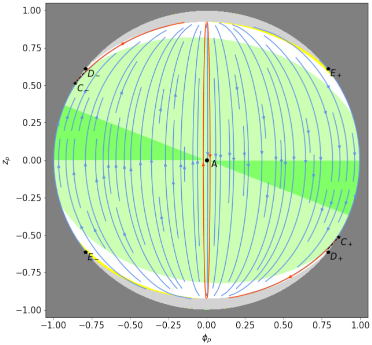
$$3FH^2 \simeq V\,, \qquad 3\dot{\phi}H = \frac{1}{m_{\text{eff}}}(-V_{\text{eff},\phi}) \tag{17}$$

  - The same scheme works for nonminimal scalar-curvature and scalar-torsion models.

What we get (nonminimal teleparallel models)

For example: nonminimal coupling $F = 1 + \xi\phi^2$,
quadratic $V = \frac{m^2}{2}\phi^2$ (left), quartic $V = \frac{\lambda}{4}\phi^4$ (right) potential.

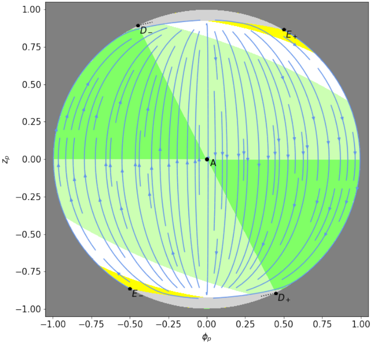


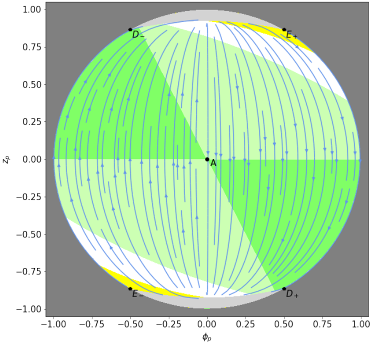


Inflation possible for very small ξ only.

What we get (nonminimal teleparallel models)

For example: nonminimal coupling $F = 1 + \xi\phi^2$,
quadratic $V = \frac{m^2}{2}\phi^2$ (left), quartic $V = \frac{\lambda}{4}\phi^4$ (right) potential.





For larger ξ the heteroclinic orbit would be in the unphysical region.

Ongoing work: Teleparallel analogue of Higgs inflation

Idea: To obtain the same phenomenology, must have the same effective potential

$$\frac{V_{Higgs}}{F_{Higgs}} = F_{tele} V_{tele} . \tag{18}$$

Approximating Higgs with a quatic potential, gives several combinations

$$F_{tele} = \frac{1}{(1+\xi\phi^2)^2} , \qquad V_{tele} = \frac{\lambda\phi^4}{4} , \tag{19}$$

$$F_{tele} = 1+\xi\phi^2 , \qquad V_{tele} = \frac{\lambda\phi^4}{4(1+\xi\phi^2)^3} , \tag{20}$$

$$F_{tele} = \xi\phi^2 , \qquad V_{tele} = \frac{\lambda\phi^2}{4\xi(1+\xi\phi^2)^2} , \tag{21}$$

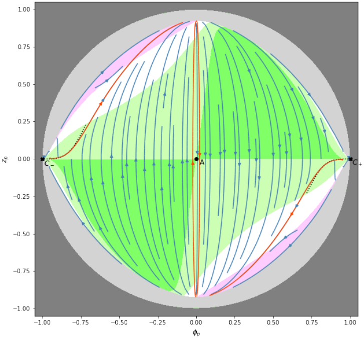
$$F_{tele} = 1 , \qquad V_{tele} = \frac{\lambda\phi^2}{4\xi(1+\xi\phi^2)^2} , \tag{22}$$

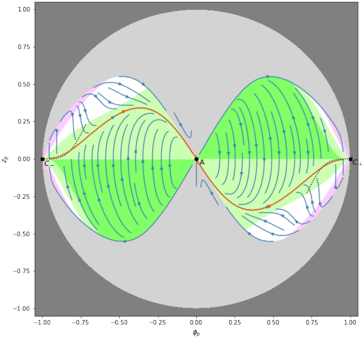
$$F_{tele} = \frac{\phi^2}{2(1+\xi\phi^2)^2} , \qquad V_{tele} = \frac{\lambda\phi^2}{2} , \tag{23}$$

For example:

$$F_{tele} = \frac{1}{(1+\xi\phi^2)^2}, \; V_{tele} = \frac{\lambda\phi^4}{4} \text{ (left)},$$

$$F_{tele} = \frac{\phi^2}{2(1+\xi\phi^2)^2}, \; V_{tele} = \frac{m\phi^2}{2} \text{ (right)}$$





But problems with perturbations, see Kolovnev, Koivisto 1808.05565; Raatikainen, Räsanen, 1910.03488.

Conclusions

- Inflation is realized by a heteroclinic orbit in the phase space (from a saddle or nonhyperbolic point to an attractor point).
- For nonminimal scalar in Jordan frame useful variables are $(\phi, \frac{\dot{\phi}}{H})$ to draw global phase portraits which
  - clearly distinguish different asymptotic regimes,
  - reveal the inflationary attractor orbit,
  - assess the correctness of slow roll approximation,
  - visualize the range of good initial conditions ($>$ 50 efolds).
- Can use the mechanical analogy with effective potential and effective mass to derive fixed points and slow roll conditions in the Jordan frame.
- Teleparallel models with quadratic nonminimal coupling do not enjoy inflation.

Järv, Toporensky 2104.10183; Järv, Lember 2104.14258.

Thinking further

- A good model of inflation needs (probably)
  - very flat effective potential,
  - an asymptotic saddle de Sitter fixed point, with the unstable or marginally unstable direction pointing into the phase space bulk,
  - a stable fixed point (for late universe),
  - the heteroclinic orbit connecting these two points running in the physical region of phase space (global portraits useful!).
- Using $V_{\rm eff}$, $m_{\rm eff}$ should be possible to construct viable models of teleparallel inflation (take some other nonminimal coupling $F(\phi)$).
- Equivalence classes of inflation models:
  - metric scalar-tensor LJ, Kannike, Marzola, Racioppi, Raidal, Rünkla, Saal, Veermäe 1612.06863
  - metric and Palatini scalar-tensor LJ, Karam, Kozak, Lykkas, Racioppi, Saal 2005.14571
  - even broader classes ???

Open problems

- Scalar-tensor: how do negative nonminimal couplings and quantum corrections alter the picture?
- Using the understanding of dynamical systems, is it possible to pinpoint the true inflationary attractor trajectory (get an analytic expression)? Get a measure how well it is approximated by the slow roll formalism.
- Reconsider the phase portraits in different sets of variables, which ones are optimal?
- A more comprehensive investigation of the generality of "good initial" conditions.
- Dynamical systems description of the whole history of the universe (from inflation to dark energy eras), like Dutta, LJ, Khyllep, Tõkke 2007.06601 .