Primordial Gravitational Waves From Cosmic Inflation

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Outline

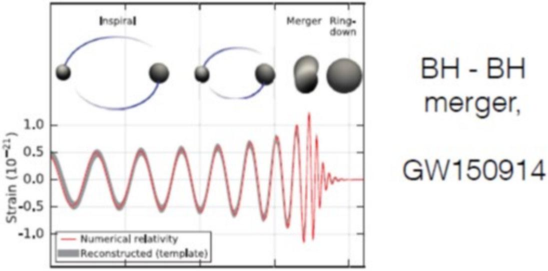
Introduction

- The direct detection of gravitational waves (GWs) by the LIGO and Virgo collaborations marked a major milestone in cosmology, launching the era of gravitational wave astronomy.
- The first observed event, originating from the merger of two $\sim 30\ M_\odot$ black holes, demonstrated the existence of astrophysical sources of GWs and opened a new window for probing the universe.
- Beyond astrophysical sources, gravitational waves can also originate from the early universe. In particular, **Primordial GWs** are a fundamental prediction of inflationary cosmology.
- These relic GWs carry imprints of the universe's earliest moments, encoding information about inflation, high-energy physics, and processes beyond the Standard Model.
- Several mechanisms can generate primordial gravitational waves:
  - Quantum fluctuations during inflation.
  - Graviton production during the reheating phase.
  - First-order phase transitions after inflation.

  These processes provide key insights into the dynamics of the early universe and the underlying physics driving its evolution.

Astrophysical Gravitational Waves

- Gravitational waves can be generated by astrophysical sources such as merging black holes, neutron stars, and other compact objects.
- These waves are produced by the acceleration and interaction of massive bodies in strong gravitational fields.
- Among the most prominent sources are binary black hole mergers, which have been observed multiple times by detectors like LIGO and Virgo.
- In the first detection, two black holes with masses of 36 $M_\odot$ and 29 $M_\odot$ merged to form a single black hole of 62 $M_\odot$. The remaining $\sim 3\ M_\odot$ was emitted as gravitational radiation.

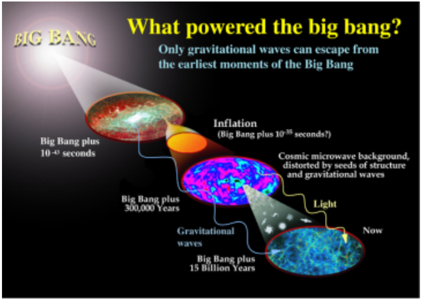


- These tiny ripples in spacetime traveled billions of years before reaching Earth, where they were finally detected, confirming a key prediction of general relativity.

Primordial & Stochastic Gravitational Waves

- **Primordial Gravitational Waves:**
  - Generated in the early universe, typically during or shortly after cosmic inflation.
  - They carry unique information about the universe's origin and fundamental physics at high energies.

- **Stochastic Gravitational Waves:**
  - Background of GWs covering the entire universe.
  - Generated by the superposition of many unresolved sources, such as cosmic strings, phase transitions, and inflationary dynamics.



- The Big Bang and cosmic inflation are key processes believed to generate stochastic gravitational waves through a variety of random, early-universe phenomena.

Gravitational Waves and Linearized Gravity

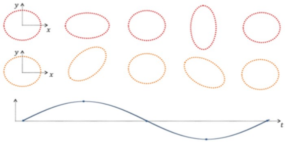
- Gravitational waves are naturally described within the framework of linearized gravity, which provides the simplest and most tractable approach to studying perturbations in an expanding universe.
- Einstein field equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$$

- Consider perturbations around a spatially flat FLRW (homogeneous and isotropic) background:

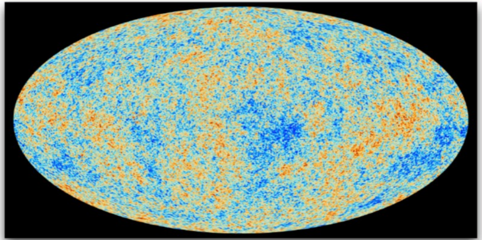
$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij}) dx^i dx^j, \quad h_{ij} \ll 1$$

- The tensor perturbation h_{ij} satisfies: $h^i_i = 0, \quad \partial_i h_{ij} = 0 \Rightarrow$ Transverse-traceless (TT) gauge $\Rightarrow$ two polarization states: $+$ and $\times$
- Equation of motion for gravitational waves: $\ddot{h}_{ij} + 3H\dot{h}_{ij} + k^2 h_{ij} = 16\pi G\, \Pi^{TT}_{ij}$
- **Homogeneous solution:** Gravitational waves sourced by quantum vacuum fluctuations $\Rightarrow$ production of gravitons in an expanding universe.
- **Inhomogeneous solution:** Gravitational waves sourced by anisotropic stress from matter fields: $\Pi^{TT}_{ij} \propto$ [scalars, vectors, fermions, tensors]



Inflation

- era of accelerated (exponential) expansion
- explains why CMB is nearly uniform



blackbody spectrum: $\bar{T}_{CMB} \sim 2.7\,K$

nearly isotropic: $\Delta T \sim 10^{-5}$

The Physics of Inflation

- Standard cosmology assumes a homogeneous and isotropic Universe, described by the Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

- Under this symmetry, the energy-momentum tensor takes the form of a perfect fluid:

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu}$$

- From Einstein's field equations, we obtain the Freedman equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

  where $H \equiv \dot{a}/a$ is the Hubble parameter.

- Inflation is defined as a phase of accelerated expansion: $\ddot{a} > 0$, which requires the pressure to satisfy $P < -\rho/3$.

- The simplest case is when $P \simeq -\rho$, leading to an approximately exponential expansion:

$$a(t) = a_\mathrm{i}\, e^{H_\mathrm{i}(t - t_\mathrm{i})}$$

- A scalar field provides a natural source of stress-energy to drive inflation. Its dynamics are governed by the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\, \partial_\mu\varphi - V(\varphi)$$

Scalar Field as a Source of Inflation

- For a scalar field φ with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - V(\varphi),$$

the corresponding stress-energy tensor is:

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} = \partial_\mu\varphi\,\partial_\nu\varphi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\varphi - V(\varphi)\right]$$

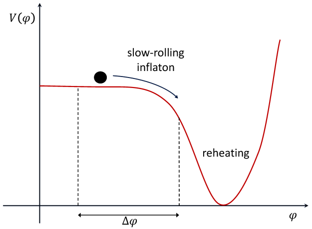
- A homogeneous scalar field $\varphi(t)$ behaves like a perfect fluid with:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

- Accelerated expansion requires $P < -\rho/3$, which is satisfied if:

$$V(\varphi) > \dot{\varphi}^2$$

- This condition is naturally achieved in regions where the potential is sufficiently flat?i.e., when the scalar field evolves slowly (slow-roll inflation).



- In such a situation, we can consider $\ddot{\varphi} \ll 3H\dot{\varphi}$, and hence

$$H^2 \simeq \frac{8\pi G}{3} V(\varphi) \,, \qquad 3H\dot{\varphi} + V_\varphi = 0$$

- The slow-roll condition is satisfied provided that

$$\frac{(V_\varphi)^2}{V} \ll H^2 \qquad \Longrightarrow \qquad \epsilon \equiv \frac{M_{\rm pl}^2}{2} \left(\frac{V_\varphi}{V}\right)^2 \ll 1 \,,$$
$$V_{\varphi\varphi} \ll H^2 \qquad \Longrightarrow \qquad \eta \equiv M_{\rm pl}^2 \frac{V_{\varphi\varphi}}{V} \ll 1 \,,$$

where ϵ and η are the so-called *slow-roll parameters*

- Successful inflation must last for a long enough period in order to solve the horizon and flatness problems.

- Typically, this feature is expressed in terms of the number of e-foldings, defined as:

$$N_{\rm tot} \equiv \int_{t_{\rm i}}^{t_{\rm f}} H \, {\rm d}t$$

where $t_{\rm i}$ and $t_{\rm f}$ are the starting and ending time of inflation, that, in case the scale-factor evolution is described by

$$N = \ln\left(a_{\rm f}/a_{\rm i}\right) ,$$

where $a_\lambda = a\left(t\left(\lambda\right)\right)$. The lower bound required to solve the horizon problem number is $N \gtrsim \ln 10^{26} \sim 60$.

Perturbations

- The observed large-scale structures in the Universe and the anisotropies in the CMB originate from small primordial fluctuations. These fluctuations evolved during the radiation and matter-dominated eras.

- Standard cosmology does not explain the origin of these perturbations. However, quantum fluctuations of the inflaton field during inflation naturally generate them, providing a robust mechanism for structure formation.

- The perturbed energy-momentum tensor for a scalar field minimally coupled to gravity is:

$$\begin{aligned} \delta T_{\mu\nu} &= \partial_\mu \delta\varphi \, \partial_\nu \bar{\varphi} + \partial_\mu \bar{\varphi} \, \partial_\nu \delta\varphi - \delta g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \bar{\varphi} \, \partial_\beta \bar{\varphi} + V(\bar{\varphi}) \right) \\ & \quad - g_{\mu\nu} \left(\frac{1}{2} \delta g^{\alpha\beta} \partial_\alpha \bar{\varphi} \, \partial_\beta \bar{\varphi} + g^{\alpha\beta} \partial_\alpha \delta\varphi \, \partial_\beta \bar{\varphi} + \frac{\partial V}{\partial \varphi} \delta\varphi \right) \end{aligned}$$

  where $\varphi = \bar{\varphi} + \delta\varphi$, with $\bar{\varphi}$ being the homogeneous background field and $\delta\varphi$ its perturbation.

- The equation of motion for the perturbed inflaton field is:

$$\delta\ddot{\varphi} + 3H \, \delta\dot{\varphi} + \frac{\partial^2 V}{\partial \varphi^2} \delta\varphi = 0$$

  where $H = \dot{a}/a$ is the Hubble parameter.

- We work in the longitudinal (Newtonian) gauge, where the metric perturbations take the form:

$$\begin{aligned} g_{00} &= -a^2(t) \left(1 + 2\psi^{(1)} + \psi^{(2)} \right) \\ g_{0i} &= 0 \\ g_{ij} &= a^2(t) \left[(1 - 2\phi^{(1)} - \phi^{(2)})\delta_{ij} + \frac{1}{2} \left(\partial_i h_j^{(2)} + \partial_j h_i^{(2)} + h_{ij}^{(2)} \right) \right] \end{aligned}$$

- Energy-Momentum at first order:

$$\begin{aligned}
\delta^{(1)} T^0_0 &= \psi^{(1)}\,\dot{\varphi}^2 - \delta^{(1)}\dot{\varphi}\,\dot{\varphi} - \delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 \\
\delta^{(1)} T^0_i &= -\,\dot{\varphi}\,\partial^i\delta^{(1)}\varphi \\
\delta^{(1)} T^i_i &= -\,\psi^{(1)}\,\dot{\varphi}_0^2 + \delta^{(1)}\dot{\varphi}\dot{\varphi}_0 \,-\, \delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 \\
\delta^{(1)} T^j_i &= 0
\end{aligned}$$

- Einstein's Tensor at first order:

$$\begin{aligned}
\delta^{(1)} G^0_0 &= 6\left(\frac{\dot a}{a}\right)^2\phi^{(1)} + 6\,\frac{\dot a}{a}\dot{\psi}^{(1)} - 2\partial_i\,\partial^i\psi^{(1)} \\
\delta^{(1)} G^0_i &= -\,2\,\frac{\dot a}{a}\partial_i\phi^{(1)} \,-\, 2\,\partial_i\dot{\psi}^{(1)} \\
\delta^{(1)} G^i_i &= 2\frac{\dot a}{a}\dot{\phi}^{(1)} + 4\,\frac{\ddot a}{a}\phi^{(1)} - 2\left(\frac{\dot a}{a}\right)^2\phi^{(1)} + \partial_i\,\partial^i\phi^{(1)} + \\
&\quad 4\,\frac{\dot a}{a}\dot{\psi}^{(1)} - \partial^i\partial_i\phi^{(1)} + \partial^i\partial_i\psi^{(1)} + 2\,\ddot{\psi}^{(1)} - \partial_i\,\partial^i\psi^{(1)} \\
\delta^{(1)} G^j_i &= -\,\partial^i\partial_j\phi^{(1)} + \partial^i\partial_j\psi^{(1)}
\end{aligned}$$

- From last equation one finds: $\phi^{(1)} = \psi^{(1)}$, and

$$\left[\ddot{\psi}^{(1)} - 2\left(2\frac{\dot a}{a} + \frac{\ddot a}{a}\left(\frac{\dot a}{a}\right)^{-1}\right)\dot{\psi}^{(1)} + \nabla^2\psi^{(1)}\right] = \kappa^2\left(\delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 + \dot{\varphi}_0\delta^{(1)}\varphi\left(\frac{\dot a}{a} + \frac{\ddot a}{a}\left(\frac{\dot a}{a}\right)^{-1}\right)\right)$$

- From these first-order perturbations of the Einstein and energy-momentum tensors, we derive a wave-like equation for the scalar metric perturbation $\psi^{(1)}$.

- This equation governs the evolution of scalar fluctuations in an expanding universe and is a cornerstone in understanding the generation of cosmic structure.

- The equation:

$$\left[\ddot{\psi}^{(1)} - 2\left(2\frac{\dot{a}}{a} + \frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)^{-1}\right)\dot{\psi}^{(1)} + \nabla^2\psi^{(1)}\right] = \kappa^2\left(\delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 + \dot{\varphi}_0\delta^{(1)}\varphi\left(\frac{\dot{a}}{a} + \frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)^{-1}\right)\right)$$

  shows that the metric perturbation $\psi^{(1)}$ is sourced by scalar field perturbations $\delta^{(1)}\varphi$, which encode quantum fluctuations during inflation.

- These fluctuations seed the curvature perturbations, which eventually give rise to the large-scale structure of the universe.

- The equality $\phi^{(1)} = \psi^{(1)}$, valid in the absence of anisotropic stress, simplifies the metric and reflects the scalar nature of inflationary perturbations.

- Through gauge-invariant variables such as the comoving curvature perturbation ($\mathcal{R} = \psi + \frac{H}{\dot{\varphi}_0}\delta\varphi$), one can track the evolution of these fluctuations and relate them to cosmological observables, like the CMB power spectrum.

Second order Perturbations

- Energy-Momentum at second order:

$$
\begin{aligned}
\delta^{(2)} T^0_0 &= -\frac{1}{2}\,\delta^{(2)}\dot{\varphi}\,\dot{\varphi}_0 - \frac{1}{2}\,\delta^{(2)}\varphi\,\frac{\partial V}{\partial \varphi}\,a^2 - \frac{1}{2}\,\left(\delta^{(1)}\dot{\varphi}\right)^2 - \frac{1}{2}\partial^i\delta^{(1)}\varphi\,\partial_i\,\delta^{(1)}\varphi \\
&- \frac{1}{2}\,\left(\delta^{(1)}\varphi\right)^2\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\,\left(\psi^{(1)}\right)^2\,\dot{\varphi}_0^2 + 2\,\psi^{(1)}\,\delta^{(1)}\dot{\varphi}\,\dot{\varphi}_0 \\
\delta^{(2)} T^0_i &= \frac{1}{2}\,\dot{\varphi}_0\,\partial^i\,\delta^{(2)}\varphi + \partial^i\,\delta^{(1)}\varphi\,\delta^{(1)}\dot{\varphi} + 2\,\dot{\varphi}_0\,\psi^{(1)}\,\partial^i\,\delta^{(1)}\varphi \\
\delta^{(2)} T^i_i &= \frac{1}{2}\,\delta^{(2)}\dot{\varphi}\,\dot{\varphi}_0 - \frac{1}{2}\,\delta^{(2)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 + \frac{1}{2}\,\left(\delta^{(1)}\dot{\varphi}\right)^2 - \frac{1}{2}\,\partial_k\,\delta^{(1)}\varphi\,\partial^k\delta^{(1)}\varphi \\
&+ 2\,\left(\psi^{(1)}\right)^2\,\dot{\varphi}_0 - \frac{1}{2}\,\left(\delta^{(1)}\varphi\right)^2\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\,\psi^{(1)}\,\delta^{(1)}\dot{\varphi}\,\dot{\varphi}_0 \\
\delta^{(2)} T^j_i &= \partial^i\delta^{(1)}\varphi\,\partial_j\delta^{(1)}\varphi
\end{aligned}
$$

Second order Perturbations

- Einstein's tensor at second order:

$$\delta^{(2)} G^0_0 = 3\frac{\dot a}{a}\dot\psi^{(2)} - \partial_i\,\partial^i\,\psi^{(2)} + \left(\frac{\dot a}{a}\right)^2\phi^{(2)} + \frac{\ddot a}{a}\,\phi^{(2)} - 12\left(\frac{\dot a}{a}\right)^2\left(\psi^{(1)}\right)^2 - 3\left(\dot\psi^{(1)}\right)^2 - 8\,\psi^{(1)}\partial_i\partial^i\psi^{(1)} - 3\,\partial_i\psi^{(1)}\partial^i\psi^{(1)}$$

$$\delta^{(2)} G^0_i = \partial^i\,\dot\psi^{(2)} + \frac{\dot a}{a}\,\partial^i\,\phi^{(2)} + \frac{1}{4}\partial_k\partial^k\left(\dot h^{i(2)}\right) + 2\,\dot\psi^{(1)}\partial^i\psi^{(1)} + 8\,\psi^{(1)}\partial^i\dot\psi^{(1)}$$

$$\delta^{(2)} G^i_i = \frac{1}{2}\,\partial_k\,\partial^k\phi^{(2)} + \frac{\dot a}{a}\dot\phi^{(2)} + \frac{\ddot a}{a}\phi^{(2)} + \left(\frac{\dot a}{a}\right)^2\phi^{(2)} - \frac{1}{2}\,\partial_k\,\partial^k\psi^{(2)} + 2\,\frac{\dot a}{a}\dot\psi^{(2)} + \dot\psi^{(2)} - 8\,\frac{\ddot a}{a}\left(\psi^{(1)}\right)^2 + 4\left(\frac{\dot a}{a}\right)^2\left(\psi^{(1)}\right)^2 - 8\,\frac{\dot a}{a}\psi^{(1)}\dot\psi^{(1)} - 3\,\partial_k\psi^{(1)}\,\partial^k\psi^{(1)} - 4\,\psi^{(1)}\,\partial_k\,\partial^k\psi^{(1)} - \left(\dot\psi^{(1)}\right)^2 - \frac{1}{2}\,\partial^i\,\partial_i\,\phi^{(2)} + \frac{1}{2}\,\partial^i\partial_i\,\psi^{(2)} + \frac{1}{2}\,\frac{\dot a}{a}\left(\partial_i\dot h^{i(2)} + \partial^i\,\dot h^{(2)}_i + \dot h^{i(2)}_{\;i}\right) + \frac{1}{4}\left(\partial_i\ddot h^{i(2)} + \partial^i\,\ddot h^{(2)}_i + \ddot h^{i(2)}_{\;i}\right) - \frac{1}{4}\,\partial_k\,\partial^k h^{i(2)}_{\;i} + 2\,\partial^i\psi^{(1)}\,\partial_i\psi^{(1)} + 4\,\psi^{(1)}\,\partial^i\partial_i\psi^{(1)}$$

$$\delta^{(2)} G^j_i = -\frac{1}{2}\,\partial^i\,\partial_j\,\phi^{(2)} + \frac{1}{2}\,\partial^i\partial_j\,\psi^{(2)} + \frac{1}{2}\,\frac{\dot a}{a}\left(\partial_j\dot h^{i(2)} + \partial^i\,\dot h^{(2)}_j + \dot h^{i(2)}_{\;j}\right) + \frac{1}{4}\left(\partial_j\ddot h^{i(2)} + \partial^i\,\ddot h^{(2)}_j + \ddot h^{i(2)}_{\;j}\right) - \frac{1}{4}\,\partial_k\,\partial^k\,h^{i(2)}_{\;j} + 2\,\partial^i\psi^{(1)}\,\partial_j\psi^{(1)} + 4\,\psi^{(1)}\,\partial^i\partial_j\psi^{(1)}$$

Gravitational Waves from Inflation

- Tensor perturbations h_{ij} represent gravitational waves, transverse, traceless fluctuations in the metric, and encapsulate two physical degrees of freedom.

- To isolate these physical tensor modes from a generic rank-2 tensor, we apply the transverse-traceless (TT) projection operator:

$$\hat{\mathcal{T}}_{ij}^{\ \ lm} = \Pi_l^i \Pi_m^j - \frac{1}{2} \Pi_{ij} \Pi^{lm}, \qquad \Pi_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\Delta}$$

  ensuring $\partial^i h_{ij} = 0$ and $h_i^i = 0$.

- Applying this operator to the second-order Einstein equations projects out the tensor dynamics:

$$\hat{\mathcal{T}}_{ij}^{\ \ lm} G_{lm}^{(2)} = \hat{\mathcal{T}}_{ij}^{\ \ lm} S_{lm}^{(2)}$$

- The evolution equation for GWs in Fourier space becomes:

$$h_{ij}{}'' + 2\mathcal{H} h_{ij}{}' + k^2 h_{ij} = 4 \hat{\mathcal{T}}_{ij}^{\ \ lm} S_{lm}$$

  where $\mathcal{H} = a'/a$ is the conformal Hubble parameter, and the left-hand side describes wave propagation with Hubble friction.

- The source term S_{lm} at second order arises from first-order scalar perturbations:

$$S_{lm} = -2\partial_l \psi^{(1)} \partial_m \psi^{(1)} - 4\psi^{(1)} \partial_l \partial_m \psi^{(1)} + \frac{1}{M_{\mathrm{Pl}}^2} \partial_l \delta\varphi^{(1)} \partial_m \delta\varphi^{(1)}$$

- These scalar sources include contributions from the gravitational potential $\psi^{(1)}$ and inflaton perturbations $\delta\varphi^{(1)}$, generating gravitational waves through anisotropic stress at second order.

Gravitational Waves Sourced by Scalar Fluctuations

- From the time-space component of Einstein's equations, we obtain:

$$\frac{\bar{\varphi}'}{\mathcal{H}}\delta\varphi = 2M_{\rm Pl}^2\left(\psi^{(1)} + \frac{\psi^{(1)\prime}}{\mathcal{H}}\right)$$

$$\Rightarrow \quad \sqrt{2\epsilon}\,\delta\varphi = 2M_{\rm Pl}\left(\psi^{(1)} + \frac{\psi^{(1)\prime}}{\mathcal{H}}\right),$$

where we used $\epsilon = -\frac{\dot{H}}{H^2}$ and $H^2 = \frac{V(\bar{\varphi})}{3M_{Pl}^2} \implies \frac{|\bar{\varphi}'|}{\mathcal{H}} = \sqrt{2\epsilon}\,M_{\rm Pl}$.

- During inflation ($\epsilon \ll 1$), the dominant contribution to the source term becomes:

$$S_{lm} \simeq \frac{1}{M_{\rm Pl}^2}\,\partial_l\delta\varphi\,\partial_m\delta\varphi$$

- Tensor perturbations are expanded as:

$$h_{ij}(\mathbf{x}) = \sum_{\lambda=+,\times}\int\frac{d^3k}{(2\pi)^3}\,e_{ij}^{\lambda}(\hat{\mathbf{k}})\,h_{\mathbf{k}}^{\lambda}\,e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, and the polarization tensors satisfy:

$$\hat{k}^i e_{ij}^{\lambda} = 0, \quad e^{\lambda\,i}_{\ i} = 0, \quad e_{ij}^{\lambda}e^{\lambda'\,ij} = \delta^{\lambda\lambda'}$$

- The equation of motion for each polarization mode becomes:

$$h_{\mathbf{k}}^{\lambda\,\prime\prime} + 2\mathcal{H}h_{\mathbf{k}}^{\lambda\,\prime} + k^2h_{\mathbf{k}}^{\lambda} = 4\,\mathcal{S}_{\mathbf{k}}^{\lambda}$$

where the source term is:

$$\mathcal{S}_{\mathbf{k}}^{\lambda} \simeq \frac{1}{M_{\rm Pl}^2}\int\frac{d^3p}{(2\pi)^3}\,e_{ij}^{\lambda}(\hat{\mathbf{k}})\,p^ip^j\,\delta\varphi_{\mathbf{p}}\,\delta\varphi_{\mathbf{k}-\mathbf{p}}$$

Power Spectrum of Gravitational Waves

- The inflaton perturbation $\delta\phi$ can be expanded in terms of creation and annihilation operators as:

$$\delta\phi(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\phi_{\mathbf{k}}(\eta)$$
$$= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[U(k,\eta)\, a(\mathbf{k}) + U^*(k,\eta)\, a^\dagger(-\mathbf{k}) \right],$$

- The power spectrum of tensor perturbations is defined via the two-point correlation function:

$$\langle h^\lambda_{\mathbf{k}}(\eta)\, h^{\lambda'}_{\mathbf{k}'}(\eta)\rangle = (2\pi)^3 \delta^{\lambda\lambda'} \delta(\mathbf{k}+\mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_h(k,\eta).$$

- Solving the tensor perturbation equation yields:

$$h^\lambda_{\mathbf{k}}(\eta) = 4 \int_{-\infty}^{\eta} d\eta'\, g_k(\eta;\eta')\, \mathcal{S}^\lambda_{\mathbf{k}}(\eta'),$$

where $g_k(\eta;\eta')$ (in conformal time η) is the Green's function satisfying:

$$g_k'' + 2\mathcal{H} g_k' + k^2 g_k = \delta(\eta-\eta').$$

- The Green's function explicit solution is given by

$$g_k(\eta;\eta') = \frac{1}{k}\frac{1}{\eta\eta'} \sin\left[k(\eta-\eta')\right]\, \Theta(\eta-\eta'),$$

Tensor Power Spectrum from Scalar Sources

- The power spectrum of tensor perturbations sourced at second order by scalar modes is given by:

$$\langle h_{\mathbf{k}}^{\lambda}(\eta)\, h_{\mathbf{k}'}^{\lambda'}(\eta)\rangle = 16\int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2\, g_k(\eta;\eta_1)\, g_{k'}(\eta;\eta_2)\,\langle \mathcal{S}_{\mathbf{k}}^{\lambda}(\eta_1)\, \mathcal{S}_{\mathbf{k}'}^{\lambda'}(\eta_2)\rangle .$$

- The two-point function of source terms is

$$\langle \mathcal{S}_{\mathbf{k}}^{\lambda}(\eta_1)\mathcal{S}_{\mathbf{k}'}^{\lambda'}(\eta_2)\rangle = \frac{1}{M_{\rm Pl}^4}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\, e_{ij}^{\lambda}(\hat{k})\, p^i p^j\, e_{mn}^{\lambda'}(\hat{k}')\, p'^m p'^n$$
$$\times \langle \delta\phi_{\mathbf{p}}(\eta_1)\delta\phi_{\mathbf{k}-\mathbf{p}}(\eta_1)\delta\phi_{\mathbf{p}'}(\eta_2)\delta\phi_{\mathbf{k}'-\mathbf{p}'}(\eta_2)\rangle .$$

- The source correlator is expressed as:

$$\langle \mathcal{S}_{\mathbf{k}}^{\lambda}(\eta_1)\mathcal{S}_{\mathbf{k}'}^{\lambda'}(\eta_2)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\delta^{\lambda\lambda'}\,\frac{\mathbf{2\pi^2}}{\mathbf{k^3}}\,\frac{\mathbf{k^4}}{\mathbf{4M_{\rm Pl}^4}}$$
$$\times \int_0^{\infty} dv \int_{|1-v|}^{|1+v|} du \left[\frac{4v^2-(1+v^2-u^2)^2}{4uv}\right]^2$$
$$\times f(uk,vk,\eta_1)f^*(uk,vk,\eta_2)\,\mathcal{P}_{\delta\phi}(uk)\,\mathcal{P}_{\delta\phi}(vk),$$

with $u \equiv |\mathbf{k}-\mathbf{p}|/k$ and $v \equiv p/k$.

- The power spectrum of the induced gravitational waves can be written as

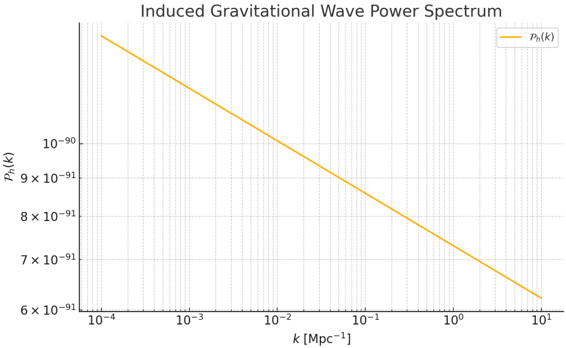
$$\mathcal{P}_h(k,\eta) = \frac{4}{M_{\rm Pl}^4} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2 - (1+v^2-u^2)^2}{4uv} \right]^2 |I(u,v,k,\eta)|^2 \, \mathcal{P}_{\delta\phi}(uk) \, \mathcal{P}_{\delta\phi}(vk),$$

where

$$I(u,v,k,\eta) \equiv k^2 \int_{-\infty}^{\eta} d\bar{\eta} \, g_k(\eta;\bar{\eta}) \; T(uk,\bar{\eta}) T(vk,\bar{\eta}).$$

- This expression captures the sourcing of second-order tensor modes h_{ij} by first-order scalar perturbations $\delta\phi$, and is ultimately determined by the scalar power spectrum $\mathcal{P}_{\delta\phi}(k)$.

- The following plot shows a numerical estimate of $\mathcal{P}_h(k)$ assuming a scale-invariant scalar spectrum and $|I|^2 \simeq 1$ (for illustrative purposes):



Is a Small $\mathcal{P}_h(k)$ Natural?

- A small tensor power spectrum $\mathcal{P}_h(k) \sim 10^{-90}$ is **natural** in the standard setup of cosmology, provided that:
  - The gravitational waves are sourced at **second order** from first-order scalar perturbations;
  - The scalar power spectrum $\mathcal{P}_{\delta\phi}(k)$ is close to the observed value $\sim 10^{-9}$;
  - There is no additional enhancement mechanism, such as resonance or sharp peaks in $\mathcal{P}_{\delta\phi}(k)$;
  - The background is a smooth, slow-roll inflationary expansion.
- In this case, the gravitational wave production is highly suppressed, and the result $\mathcal{P}_h(k) \ll 1$ simply reflects the weak sourcing of tensor modes by typical scalar perturbations in slow-roll inflation.
- There are scenarios where the amplitude of $\mathcal{P}_h(k)$ can be significantly enhanced:
  - If the scalar power spectrum $\mathcal{P}_{\delta\phi}(k)$ is amplified at small scales (e.g., in models with sharp features or ultra-slow-roll inflation);
  - In scenarios where **primordial black holes (PBHs)** are produced, requiring enhanced scalar fluctuations ? which in turn source observable GWs;
  - Through non-standard mechanisms like preheating, resonant amplification, or spectator fields;
  - If first-order tensor perturbations from inflation are large enough (i.e., large tensor-to-scalar ratio r), they may be detectable by CMB B-mode experiments.

Gravitational Wave Production

- The present-day energy density of gravitational waves is given by:

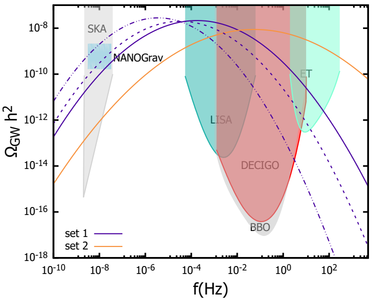
$$\Omega_{\mathrm{GW}}(k) = \frac{1}{24}\left(\frac{k}{aH}\right)^2 \overline{\mathcal{P}_h(\tau, k)},$$

  where the overline denotes the time average.
- For induced gravitational waves sourced by scalar perturbations, we have:

$$\Omega_{\mathrm{GW}}(k) = \frac{c_g\,\Omega_r}{36}\int_0^{1/\sqrt{3}} \mathrm{d}d \int_{1/\sqrt{3}}^{\infty} \mathrm{d}s \left[\frac{(s^2-1/3)(d^2-1/3)}{s^2+d^2}\right]^2 \mathcal{P}_{\mathcal{R}}(kx)\,\mathcal{P}_{\mathcal{R}}(ky)\,(I_c^2+I_s^2),$$

  where $\Omega_r = 5.4\times10^{-5}$ and $c_g = 0.4$.

- $\mathcal{P}_{\mathcal{R}}(k_*) \simeq 2.1\times10^{-9}$, at $k_* = 0.05\,\mathrm{Mpc}^{-1}$. In scenarios with enhanced small-scale perturbations (e.g., PBH formation), $\mathcal{P}_{\mathcal{R}}$ can reach much larger values: $\mathcal{P}_{\mathcal{R}}(k_{\mathrm{peak}}) \sim 10^{-2} - 10^{-1}$.
- The predicted GW spectra for these parameter choices lie within the sensitivity ranges of future GW detectors such as LISA, DECIGO, BBO, SKA, and ET.
- The recently reported NANOGrav signal may be interpreted within this model, as shown by the purple lines in the plot.



Gravitational Waves During Reheating

- We consider a non-instantaneous reheating scenario in which the inflaton decays exclusively into right-handed neutrinos (RHNs).
- The subsequent decay of RHNs into Standard Model particles not only reheats the Universe but also generates the observed baryon asymmetry via leptogenesis.
- During this phase, gravitational waves (GWs) are produced through two primary channels: bremsstrahlung radiation during inflaton decay and inflaton scattering processes.
- GW production from bremsstrahlung becomes dominant toward the end of reheating, whereas scattering-induced GWs are more prominent around the maximum temperature of the Universe.
- The resulting GW spectrum from both processes lies within the sensitivity range of proposed resonant cavity experiments.

Inflaton Graviton Interactions During Reheating

- The interaction between the inflaton field ϕ and gravity is described by the action:

$$S = \int d^4x \sqrt{|g|} \left[\frac{M_P^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],$$

  where $V(\phi)$ denotes the inflaton potential.

- In the weak-field limit, the metric is expanded as:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha^{\ \nu}, \\ \sqrt{|g|} &= 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8} (h^2 - 2h^\mu_\alpha h^\alpha_\mu), \end{aligned}$$

  with $\kappa = \sqrt{2}/M_P$.

- The leading inflatongraviton interaction term is:

$$\mathcal{L}_{\text{int}} \supset -\kappa \, h_{\mu\nu} \, T_\phi^{\mu\nu},$$

  where the inflaton energy-momentum tensor is:

$$T_\phi^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \left[\frac{1}{2} (\partial\phi)^2 - V(\phi) \right].$$

- During reheating, the inflaton oscillates around a generic potential:

$$V(\phi) = \lambda M_P^4 \left(\frac{\phi}{M_P} \right)^n.$$

Gravitational Waves from Inflaton Decay

- Gravitational waves (GWs) can be sourced during reheating via the decay of the inflaton field ϕ into matter fields, accompanied by graviton emission.

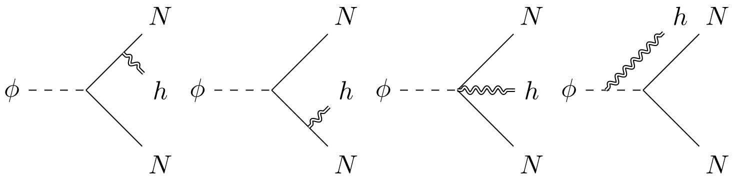
- We consider generic interactions of the inflaton with a complex scalar χ, Dirac fermion ψ, and massive vector boson V_μ:

$$\mathcal{L}_{\rm int} \supset -\mu\phi|\chi|^2 - y_\psi\bar{\psi}\psi\phi - g_V V_\mu V^\mu \phi.$$

- After inflation ends, the inflaton oscillates around the minimum of its potential and decays into particlese.g., heavy right-handed neutrinos (RHNs)with possible graviton emission:

$$\phi \to N\bar{N}, \quad \phi \to h\,N\bar{N}.$$

- The emitted gravitons contribute to a stochastic gravitational wave background, encoding information about inflaton interactions and reheating dynamics.



Inflaton Decay into RHNs and GW Emission

- The tree-level decay rate of the inflaton into RHNs is:

$$\Gamma^{(0)} = \frac{M}{8\pi} \left(\frac{y_\phi}{\sqrt{2}} \right)^2 \left(1 - \frac{4m_N^2}{M^2} \right)^{3/2},$$

where M is the inflaton mass, m_N is the RHN mass, and y_ϕ is the Yukawa coupling.

- The differential rate for graviton emission in the 3-body decay $\phi \to h\, N\bar{N}$ is:

$$\frac{d\Gamma^{(1)}}{dE_h} = \frac{y_\phi^2}{32\pi^3 M_P^2} \cdot \frac{E_h^3}{M^2} \cdot \left(1 - \frac{4m_N^2}{(M - E_h)^2} \right)^{1/2} \cdot \left(1 - \frac{2E_h}{M} \right)^2,$$

valid for $0 < E_h < M/2 - m_N$.

- The energy densities evolve as:

$$\begin{aligned}
\dot{\rho}_\phi + 3H\rho_\phi &= -(\Gamma^{(0)} + \Gamma^{(1)})\rho_\phi, \\
\dot{\rho}_N + 3H\rho_N &= \Gamma^{(0)}\rho_\phi + \int \frac{d\Gamma^{(1)}}{dE_h} \left(\frac{M - E_h}{M} \right) \rho_\phi \, dE_h - \Gamma_N \rho_N, \\
\dot{\rho}_R + 4H\rho_R &= \Gamma_N \rho_N, \\
\dot{\rho}_{GW} + 4H\rho_{GW} &= \int \frac{d\Gamma^{(1)}}{dE_h} \left(\frac{E_h}{M} \right) \rho_\phi \, dE_h,
\end{aligned}$$

with $H = \frac{\rho_\phi + \rho_N + \rho_R + \rho_{GW}}{3M_P^2}$.

- One can write the last Boltzmann equation as:

$$\frac{d(\rho_{GW}/\rho_R)}{da} \simeq \frac{1}{aH}\frac{\rho_\phi}{\rho_R}\left[\int \frac{d\Gamma^{(1)}}{dE_w}\frac{E_w}{M}dE_w - \frac{\rho_{GW}}{\rho_R}\Gamma^{(0)}\right]$$

This expression can be integrated during reheating, *i.e.*, $a_{\rm max} \leq a \leq a_{rh}$, corresponding to photon temperature $T_{\rm max} \geq T \geq T_{rh}$.

- During reheating $\rho_\phi(a) = \rho_\phi(a_{rh})\left(\frac{a_{rh}}{a}\right)^3$, and T_{rh} corresponds to $\rho_R(T_rh) = \rho_\phi(T_{rh})$.

- Assuming that at the beginning of the reheating, no SM radiation or GWs and at the end of the reheating $\Gamma^{(0)} \simeq H(T_{rh})$. Thus, one finds for $y \to 0$ that

$$\frac{\rho_{GW}(T_{rh})}{\rho_R(T_{rh})} \simeq \frac{1}{96}\frac{M^2}{\pi^2 M_P^2}\left[1 - (\frac{T_{rh}}{T_{\rm max}})^{8/3}\right].$$

- The primordial GW spectrum at present $\Omega_{GW}(f)$ for a frequency f is defined by

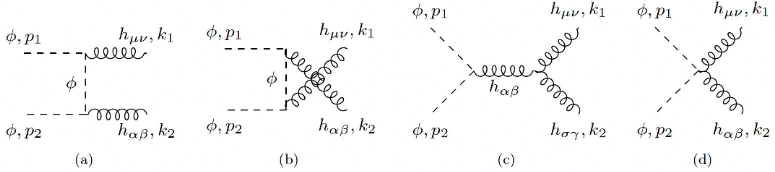
$$\Omega_{GW}(f) = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln f} = \Omega_\gamma^{(0)}\frac{d(\rho_{GW}/\rho_R)}{d\ln f} = \Omega_\gamma^{(0)}\frac{g_*(T_{rh})}{g_*(T_0)}\left[\frac{g_{*s}(T_0)}{g_{*s}(T_{rh})}\right]^{4/3}\frac{d(\rho_{GW}(T_{rh})/\rho_R(T_{rh})}{d\ln E_w}$$

$$\Rightarrow \qquad \Omega_{GW}(f) \simeq 1.4\times 10^{-8}\frac{T_{rh}}{5.5\times 10^15 \mathrm{GeV}}\frac{M}{M_P}\frac{f}{10^{12}\mathrm{Hz}}.$$

GW Production During Reheating

- Gravitational waves can also be produced from inflaton self-scattering during reheating.
- A quartic interaction $hh\phi\phi$ arises at second order, with coupling κ^2, leading to $\phi\phi \to hh$ processes.
- These processes are dominant at the onset of reheating and are proportional to the square of the inflaton energy density.
- They depend only on gravitational couplings and not on inflaton decay channels.
- All four contributing Feynman diagrams enter the matrix element $\mathcal{M}(\phi\phi \to h_{\mu\nu}h_{\mu\nu})$ at $\mathcal{O}(k^2)$.
- The GW production rate:

$$\Gamma_h = \frac{\rho_\phi}{m_\phi} \cdot \frac{\sum_{\text{pol}} |\mathcal{M}|^2}{32\pi m_\phi^2}.$$



GW Spectrum from Inflaton Condensate

- The Boltzmann equations governing GW production from inflaton scattering are:

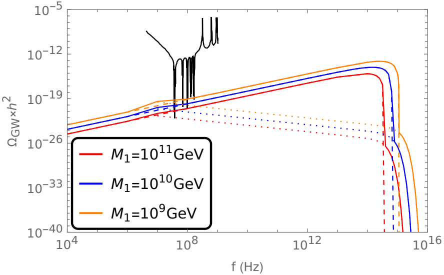
$$\dot{\rho}_{GW} + 4H\rho_{GW} = \Gamma_h \rho_\phi,$$
$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi,$$
$$\dot{\rho}_R + 4H\rho_R = \Gamma_\phi \rho_\phi.$$

- Solving yields:

$$\rho_{GW} = \frac{\sqrt{3}\alpha^{4/3} m_\phi \rho_{\rm end}^{1/6} T_{RH}^{16/3}}{16\pi M_P^3},$$

with $\rho_{\rm end}$: inflaton energy density at $a = a_{\rm end}$, and $\alpha = \frac{g_{RH}\pi^2}{30}$.

- The GW energy density today is: $\Omega_{GW} h^2 = \left(\frac{d\rho_{GW}}{d\ln f}\right) / (\rho_{c,0} h^{-2})$, and GW frequency today from scale factor a: $2\pi f = m_\phi (a/a_0)$.



- Example spectrum from inflaton annihilation for $m_\phi = 3 \times 10^{13}$ GeV and $\rho_{\rm end} = (5.5 \times 10^{15}\ \text{GeV})^4$.

Conclusion and Outlook

- Gravitational wave (GW) production provides a powerful probe of inflationary dynamics and the early Universe.
- We discussed two primary mechanisms for GW production:
  - **Inflationary Fluctuations:** Quantum fluctuations during inflation generate a background of GWs, which encode the properties of the inflationary phase.
  - **Reheating Processes:** After inflation, GW production occurs via:
    - **Graviton Bremsstrahlung** during inflaton decay, sensitive to couplings with matter fields.
    - **Inflaton scattering/annihilation**, purely gravitational, independent of decay channels.
- Both mechanisms contribute to a high-frequency stochastic GW background, potentially detectable by future observatories.
- The GW spectra carry valuable information on:
  - The properties of inflation (potential, scale, and dynamics)
  - Reheating temperature and the inflaton mass
  - Couplings to other fields
- Current detectors are not sensitive to the high-frequency GWs from these processes, but upcoming advancements may allow detection.
- **Outlook:** Gravitational wave cosmology offers a complementary and novel probe of the early Universe, shedding light on inflation, reheating, and physics beyond the Standard Model.