Summer School and Internship 2022 CTP - BUE (3 - 21 July 2022)

Hamiltonian formalism of GR and numerical relativity - Lecture 2

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Outline

- Lecture 1. Hamiltonian formalism in physics
- Lecture 2. ADM formulation of General Relativity
- Lecture 3. Basics of Numerical Relativity

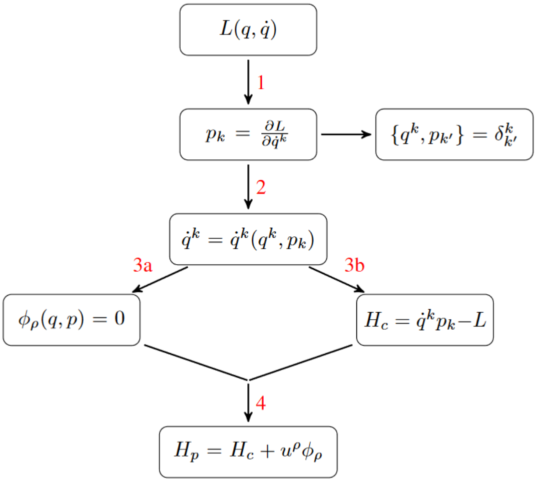
References

- K. Sundermeyer, "Symmetries in Fundamental Physics" (2014) [Appendix C]
- Eric Gourgoulhon, 3+1 Formalism in General Relativity: Bases of Numerical Relativity (2012) [arXiv:gr-qc/0703035]
- Eric Poisson, A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics (2004)
- Strong Field Gravity (East) - Perimeter Institute for Theoretical Physics course PSI 2018/2019
- T. Baumgarte and S. Shapiro, "Numerical Relativity: Solving Einstein's Equations on the Computer", (2010)

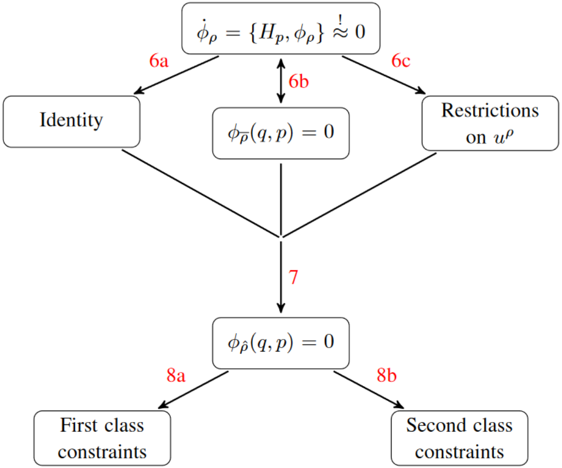
Overview of previous lecture

- The study of numerical relativity is well motivated from the novel detection of binary black hole mergers.
- Predicting their behaviour is not possible to be done fully analytically, due to nonlinearity of Einstein's equations and strong gravity regime.
- We reviewed Dirac algorithm for constrained Hamiltonian systems.

Summary of Dirac algorithm (1/2)



Summary of Dirac algorithm (2/2)



Hypersurfaces

- Spacetime is considered as a differential manifold $\mathcal{M}$ of dimension 4 equipped with a metric g of signature $(-,+,+,+)$.
- The couple $(\mathcal{M}, g)$ is called a Lorentzian manifold. It is assumed that $\mathcal{M}$ is time orientable.
- We consider a 3-dimensional manifold $\hat{\Sigma}$ and an embedding $\Phi_* : \hat{\Sigma} \rightarrow \mathcal{M}$. The image $\Sigma = \Phi_*(\hat{\Sigma}) \subset \mathcal{M}$ is called a hypersurface of $\mathcal{M}$.

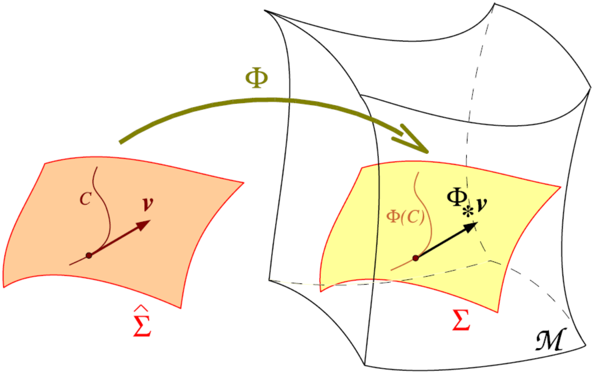


Figure: Embedding Φ of the 3-D manifold $\hat{\Sigma}$ into the 4-D manifold $\mathcal{M}$, defining the hypersurface $\Sigma = \Phi(\hat{\Sigma})$. The pushforward $\Phi_{\star} v$ of a vector v tangent to some curve C in $\hat{\Sigma}$ is a vector tangent to $\Phi(C)$ in $\mathcal{M}$. The pullback $\Phi^{\star}$ has the opposite action than the pullforward. [Taken from E. Gourgoulhon (2007)]

Hypersurfaces

- The pullback of the metric g is called the **first fundamental form** or **induced metric**, or 3-metric of Σ

$$\gamma = \Phi^*(g): \quad T_p(\Sigma) \times T_p(\Sigma) \longrightarrow \mathbb{R} \tag{1}$$

- Depending on the signature of γ, Σ is called
  (i) **spacelike**, if γ is Riemannian with signature $(+,+,+)$
  (ii) **timelike**, if γ is Lorentzian with signature $(-,+,+)$
  (iii) **null**, if γ is degenerate
- From now on, γ will be regarded as spacelike.
- γ has a unique Levi-Civita connection. This implies a unique covariant derivative D.

Normal vector

- A hypersurface can be locally defined as the set of points for which a scalar field t defined on $\mathcal{M}$ is constant, also called the **time function**.
- It is possible to define a covector ω_μ orthogonal to Σ, therefore denoting the normal to the leaves of the foliation

$$\omega_\mu = \nabla_\mu t. \tag{2}$$

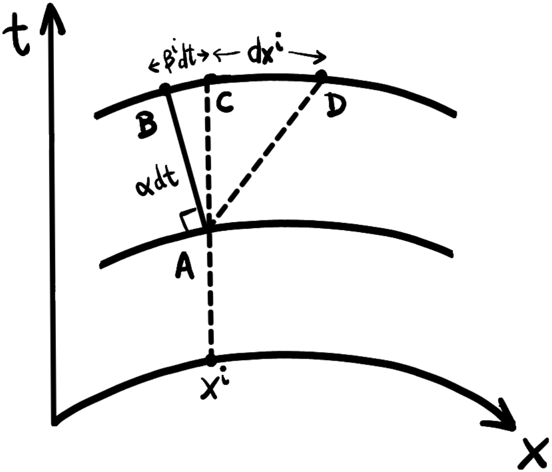
- The covector ω_μ is closed, that is

$$\nabla_{[\mu}\omega_{\nu]} = \nabla_{[\mu}\nabla_{\nu]}t = 0. \tag{3}$$

- We define the scalar α called the **lapse function** as

$$g^{\mu\nu}\nabla_\mu t \nabla_\nu t = \nabla^\mu t \nabla_\mu t = -1/\alpha^2. \tag{4}$$

Lapse and shift functions



$A = (t, x^i)$

From A we project the normal vector landing at point B, but it is shifted w.r.t. original x^i by $\beta^i dt$

$C = (t+dt, x^i)$

ADM metric represents the distance between A and

$D = (t+dt, x^i + dx^i)$

Lapse and shift functions

- The distance between A and B is not dt but the proper time $d\tau$. To account for this difference we introduce the **lapse function** α such that $d\tau = \alpha dt$.
- The normal vector is orthogonal to the hypersurface $t =$ const. where A belongs. But its intersection with the hypersurface $t + dt$ lands in B, which might not necessarily have the original spatial coordinates x^i of A. To account for this difference, we introduce the **shift function** β^i such that the coordinates of B are $(t + dt, x^i - \beta^i dt)$.
- Distances between B, C and D are measured by the spatial metric γ_{ij}. The line element between B and D is $\gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$
- Considering proper signature, the line element between A and D is

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{5}$$

This defines the Arnowitt, Deser & Misner (ADM) metric.

The metric of the foliated spacetime can be read from the previous line element, and corresponds to

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \gamma_{ij}\beta^i\beta^j & \beta^i \\ \beta^j & \gamma_{ij} \end{pmatrix}, \tag{6}$$

while its inverse is

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}. \tag{7}$$

It will also be useful to define the normal vector in coordinates such that

$$n_\mu = (-\alpha, \vec{0}), \quad n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), \tag{8}$$

where we have raised the index in n^μ with the inverse metric $g^{\mu\nu}$.

The intrinsic metric

- The spacetime $g_{\mu\nu}$ induces a 3D Riemannian metric γ_{ij} on each hypersurface.
- This intrinsic metric is given by

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{9}$$

- $\gamma_{\mu\nu}$ is purely spatial, it has no components along n^μ.
- Contracting with the normal

$$n^\mu \gamma_{\mu\nu} = n^\mu g_{\mu\nu} + n_\mu n^\mu n_\nu = n_\nu - n_\nu = 0. \tag{10}$$

- The inverse 3-metric $\gamma^{\mu\nu}$ is obtained by raising the indices

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu. \tag{11}$$

The intrinsic metric

- The 3-metric $\gamma_{\mu\nu}$ can be used to project geometric objects along the direction of n^μ.
- Given a tensor $T_{\mu\nu}$, its spatial part $T^{\perp}_{\mu\nu}$ is defined as

$$T^{\perp}_{\mu\nu} = \gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma} T_{\rho\sigma}. \tag{12}$$

- A normal projector can be defined as $N_{\mu}{}^{\nu} = -n_\mu n^\nu = \delta^\nu_\mu - \gamma_\mu{}^\nu$, such that any arbitrary vector can be decomposed as

$$v^\mu = \delta^\mu_\nu v^\nu = (\gamma^\nu_\mu + N^\mu_\nu)v^\nu = v^{\perp\mu} - n^\mu n_\nu v^\nu. \tag{13}$$

Covariant derivatives on hypersurfaces

- The 3-metric γ_{ij} defines univocally a covariant derivative D_i with the Levi-Civita associated to γ_{ij}.
- We can extend it to a 4D spacetime as D_μ, compatible with the metric $\gamma_{\mu\nu}$. For example, for scalars

$$D_\mu \phi \equiv \gamma^\nu_\mu \nabla_\nu \phi, \tag{14}$$

and for a $(1,1)$ tensor, it works as

$$D_\mu T^\nu{}_\rho \equiv \gamma^\alpha_\mu \gamma^\nu_\beta \gamma^\gamma_\rho \nabla_\alpha T^\beta{}_\gamma \tag{15}$$

Covariant derivatives on hypersurfaces

- We can harmlessly associate a curvature tensor ${}^{(3)}R^{\alpha}{}_{\beta\mu\nu}$ to the previous covariant derivative

$$D_\mu D_\nu v^\alpha - D_\nu D_\mu v^\alpha = {}^{(3)}R^{\alpha}{}_{\beta\mu\nu} v^\beta, \tag{16}$$

  where it can also be proved that ${}^{(3)}R^{\alpha}{}_{\beta\mu\nu} n^\beta = 0$.
- A Ricci tensor and scalar can also be defined

$$ {}^{(3)}R_{\beta\nu} = {}^{(3)}R^{\alpha}{}_{\beta\alpha\nu}, \quad {}^{(3)}R = g^{\mu\nu}\,{}^{(3)}R_{\mu\nu}. \tag{17}$$

Extrinsic curvature

- Einstein's field equations impose some conditions on ${}^{(4)}R^{\rho}{}_{\sigma\mu\nu}$. In order to understand its effect on hypersurfaces we need to decompose it into spatial parts.
- This decomposition will naturally include ${}^{(3)}R^{\rho}{}_{\sigma\mu\nu}$. It measures the **intrinsic curvature** on a hypersurface Σ_t. It has no information about how the hypersurface is fitted in the 4D manifold.
- The missing information is contained in the **extrinsic curvature**.

Extrinsic curvature

- The symmetric rank-2 tensor

$$K_{\mu\nu} = -\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}\nabla_{(\alpha}n_{\beta)} = -\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}\nabla_{\alpha}n_{\beta} \tag{18}$$

  is called the **second fundamental form** or **extrinsic curvature** of the hypersurface Σ.
- It is denoted as *extrinsic* since it is defined in terms of n and not in $T(\Sigma)$.
- By construction, it is symmetric and purely spatial. It measures how the normal to the hypersurface changes from point to point.
- It also measures the rate at which the hypersurface deforms as it is carried along the normal.

Example 1: The plane

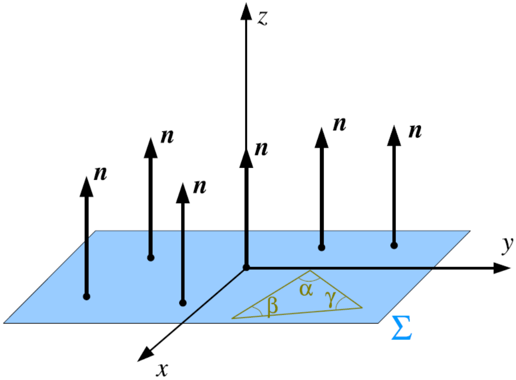


Figure: Consider the plane as a hypersurface in $\mathbb{R}^3$. The normal vector *n* stays constant along it, therefore the extrinsic curvature of Σ vanishes. The intrinsic curvature of Σ, γ vanishes as well. [Taken from E. Gourgoulhon (2007)]

Example 2: The cylinder

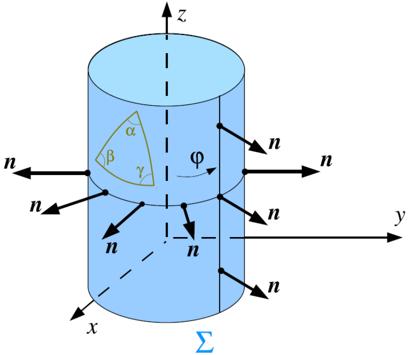


Figure: In a cylinder as hypersurface of $\mathbb{R}^3$, the normal vector stays constant when z varies at fixed φ, but changes direction as φ varies at fixed z. It means that the extrinsic curvature of Σ is nonzero only in the φ direction. The intrinsic curvature is identically zero. [Taken from E. Gourgoulhon (2007)]

Example 3: The sphere

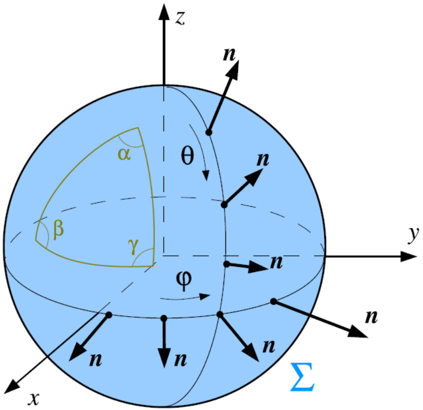


Figure: On a sphere, the extrinsic curvature is nonvanishing along both θ and φ directions. Moreover, the intrinsic curvature does not vanishes as well, as the sum of angles of any triangle is $\alpha + \beta + \gamma > \pi$. [Taken from E. Gourgoulhon (2007)]

Acceleration

- A useful concept is the **acceleration** of a foliation

$$a_\mu = n^\nu \nabla_\nu n_\mu. \tag{19}$$

- The extrinsic curvature can be written in terms of it as

$$\begin{aligned}
K_{\mu\nu} &= -\gamma^\alpha_\mu \gamma^\beta_\nu \nabla_\alpha n_\beta &(20)\\
&= -(\delta^\alpha_\mu + n^\alpha n_\mu)(\delta^\beta_\nu + n^\beta n_\nu)\nabla_\alpha n_\beta &(21)\\
&= -(\delta^\alpha_\mu + n^\alpha n_\mu)\delta^\beta_\nu \nabla_\alpha n_\beta &(22)\\
&= -\nabla_\mu n_\nu - n_\mu a_\nu. &(23)
\end{aligned}$$

Extrinsic curvature

- The Lie derivative of a generic tensor $T_{\mu\nu}$ along a vector v^μ is expressed as

$$\mathcal{L}_v T_{\mu\nu} = v^\rho \partial_\rho T_{\mu\nu} + (\partial_\mu v^\rho) T_{\rho\nu} + (\partial_\nu v^\rho) T_{\mu\rho}. \tag{24}$$

- An alternative expression for the extrinsic curvature is in terms of the Lie derivative along n^μ. It can be proved that

$$\mathcal{L}_n \gamma_{\mu\nu} = -2K_{\mu\nu}. \tag{25}$$

- An important definition for later is the **mean curvature**

$$K \equiv g^{\mu\nu} K_{\mu\nu} = \gamma^{\mu\nu} K_{\mu\nu}. \tag{26}$$

The Gauss-Codazzi equation

- We first compute the double covariant derivative acting on any vector

$$D_\mu D_\nu v^\rho = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\gamma \nabla_\alpha \nabla_\beta v^\gamma - K_{\mu\nu}\gamma^\rho_\alpha n^\beta \nabla_\beta v^\alpha - K^\rho_\mu K_{\nu\alpha} v^\alpha. \tag{27}$$

- We recall the definition of the intrinsic curvature

$$D_\mu D_\nu v^\alpha - D_\nu D_\mu v^\alpha = {}^{(3)}R^\alpha{}_{\beta\mu\nu} v^\beta, \tag{28}$$

and combine all previous equations to obtain the **Gauss-Codazzi equation**

$${}^{(4)}R^\rho{}_{\sigma\mu\nu}\gamma^\mu_\alpha \gamma^\nu_\rho \gamma^\gamma_\rho \gamma^\sigma_\delta = {}^{(3)}R^\gamma{}_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\alpha\delta}. \tag{29}$$

- This equation relates the spatial projection of the spacetime curvature tensor to the 3D curvature.

Codazzi-Mainardi equation

- Another identity can be derived from considering projections of the Riemann tensor along the normal direction. It involves a spatial derivative of the extrinsic curvature.
- The covariant derivative of the extrinsic curvature corresponds to

$$D_\mu K_{\nu\rho} = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho \nabla_\alpha K_{\beta\gamma}, \tag{30}$$

  and after subtracting an index permutation it is obtained

$$D_\nu K_{\mu\rho} - D_\mu K_{\nu\rho} = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho n^{\delta\,(4)} R_{\alpha\beta\gamma\delta}, \tag{31}$$

  which is the so-called Codazzi-Mainardi equation.

The constraint equations

- The $3+1$ decomposition of Einstein's equations allows to identify the **intrinsic metric** $\gamma_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ of an initial hypersurface as the **initial data** to be prescribed for the evolution equations of GR.
- We will see that not all the components of γ and K freely propagate, as there are constraints hidden in the Gauss-Codazzi and Codazzi-Mainardi equations. We will assess them by considering the Einstein's equations in vaccuum ${}^{(4)}R_{\mu\nu} = 0$.