Misner string:

TN metric has a string-like singularity along z-axis to see that let us consider a family of time-like surface generated by surface $f = t$, which gives $\Rightarrow$ $\nabla_\mu t = \xi_\mu$. Its norm is

$$|\xi|^2 = |\nabla t|^2 = \frac{4n^2}{(r^2-n^2)} \frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{f(r)}.$$

* This shows two singular behaviors, one at $f(r)=0$, which is a coordinate singularity and at $\theta = 0, \pi$ which can not be removed at the same time using proper coordinates! Metric is

$$ds^2 = -f(dt - 2n(\cos\theta + c)\,d\varphi)^2 + f^{-1}dr^2 + \cdots$$

set $c=-1$.

$$ds^2_{(+)} = -f(dt - 2n(\cos\theta - 1)\,d\varphi)^2 + \cdots$$

set $c=+1$.

$$ds^2_{(-)} = -f(dt - 2n(\cos\theta+1)\,d\varphi)^2 + \cdots$$

* Two patches can cover mnfld smoothly only if

$$t^+ = t^- - 4n\varphi$$

$$\Rightarrow \quad \Delta t^{(\pm)} = 4n \overbrace{\Delta\varphi}^{2\pi} \Rightarrow \boxed{\beta = 8\pi n.}$$

* But for our soln. $\beta^{(bh)} = 4\pi r_0 = 8\pi n \Rightarrow \boxed{r_0 = 2n}$ $\boxed{M = \frac{3}{4}n.}$

* This bring restriction to thermodynamics which affect the existence of independent terms like $\Phi_n\, dn$ which could contribute to BH thermodynamics!

TN unconstrained Thermodynamics:

* The prize for removing Misner string is constraining thermodynamics
$$\beta(r_0) = 8\pi n.$$
* As a result entropy $S \not\propto A_h$ (not horizon area)
* mass has to be $M = \frac{3}{4} n$
* $T \xrightarrow{n \to 0} \neq T_{sch}$ (it diverges in this limit!)
* This situation lead a group of authors to relax the above cond.
* Their idea was to add a pair of thermody. quantity (X, Y) such that
$$dU = T dS + \Phi_e dQ_e + \underbrace{X dY}$$
(two possible interpretations)

① R. Mann
$$X dY = T_{MS} d S_{MS}$$
but this $\Rightarrow$ $\beta_{MS} = \beta = 8\pi n$
(back to □ 1.)

②
$$X dY = \phi_N dN$$
where N is a conserved quant.

* We proposed the second choice with $N = n$ and $\phi_N = -\frac{n}{2r_0}$. This is true in TN-Minkowski
* for TN-AdS $N = n\left(1 + \frac{4n^2}{\ell^2}\right)$, $\Lambda = -3/\ell^2$, (ℓ AdS radius)
* In the coming pages we will talk more about TN-dyons in Minkowski space.

Conserved charges for TN – Mink:

* To calculate total mass we use Komar' integral with $\chi^{(t)}$,

$$\left(-\frac{1}{4\pi}\right)\left(*d\chi^{(t)}\right)=\frac{1}{4\pi}\left(\frac{4n^2f}{r^2+n^2}\cos\theta\, dr\wedge d\varphi+f'(r^2+n^2)\sin\theta\, d\theta\wedge d\varphi\right)$$

Masses: $M^{\infty}=M$ & $M^{(+)}=M^{(-)}=-n\phi_n$

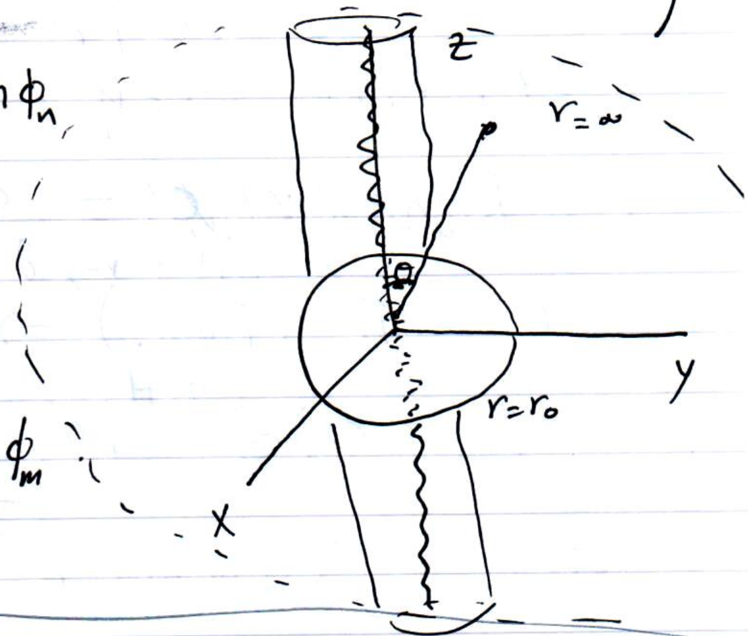
$M^h=M-2n\phi_n$

NUT charge: $N^{\infty}=n$ & $N^h=0$

$N^+=N^-=\frac{n}{2}$.

E.M. charges: $Q_e^{\infty}=q_e$ & $Q_e^h=q-2n\phi_m$

$Q_m^{\infty}=q_m$ & $Q_m^h=q_m+2n\phi_e$



* Important: There exist charges along z-axis and they are indep of having Misner string or not!

* 1st law $d\left(\overbrace{M-n\phi_n}^{U}\right)=TdS+\phi_n dn+\phi_e dQ_e^{\infty}+\phi_m dQ_m^h$.

* internal energy is not the mass, but $U=M-n\phi_n>0$, which is telling us that there is another source for the total energy which is the nut charge besides the mass "M"!

* On-shell gravit. action $I_0=I(T,n)=\beta \underset{\text{Helmholtz free Energy}}{F}$

$F=\frac{I}{\beta}=U-TS.$

* To understand more about BH thermodynamics and their ensembles we need to discuss Euclidean Path integral since it is the formal Theory for calculating those quantities.

In Q.M. Path integral

(q_2, t_2)

(q_1, t_1)

$$\langle q_2(t_2)|q_1(t_1)\rangle = \int D[q]\, e^{iI(q)}$$

In Q.F.T

$$= \langle \phi_2, t_2|\phi_1, t_1\rangle = \int D[\phi]\, e^{iI(\phi)}$$

$$= \langle \phi_2| e^{-iH(t_2-t_1)}|\phi_1\rangle$$

$t_2 = t,\ t_1 = 0,\ g = \eta,$

$$Z = \langle \phi_2| e^{-iHt}|\phi_1\rangle$$

arbitrary ?

$t = -i\beta$ (Eucl.) $\Rightarrow$ $\phi_1 = \phi_2$

$$\sum_\phi \langle\phi|e^{-\beta H}|\phi\rangle \overset{E}{=} \mathrm{tr}\, e^{-\beta H} = Z_{\text{part.}}$$

$$Z = \int D[g,\phi]\, e^{-I^E(g,\phi)} = \mathrm{tr}\, e^{-\beta H(g,\phi)}$$

In gravity β is not arb. since regularity of the fields at boundary requires certain temp.!

$$ds^2 = -f\,dt + f^{-1}dr^2 + r^2 d\Omega^2 \ ; \quad f = 1 - \frac{2M}{r}$$

$$f = f(r_0)^{0} + (r-r_0) f' + \cdots$$

$$ds^2 = +(r-r_0) f'\, d\tau + \frac{dr^2}{(r-r_0)f'} + \cdots$$

$$\int dR = \int \frac{dr}{\sqrt{(r-r_0)f'(r_0)}} \Rightarrow R = \frac{2}{\sqrt{f'}}\sqrt{(r-r_0)} \Rightarrow \frac{R^2}{4} f' = (r-r_0)$$

$$ds^2 = \frac{R^2}{4} f'^2 d\tau^2 + dR^2 + \cdots \qquad \Delta\tau = \beta = T^{-1}!$$

$$R^2 d\phi^2 + dR^2$$

$$R^2 d\left(\frac{f'}{2}\tau\right)^2 \Rightarrow 2\pi = \frac{f'}{2}\Delta\tau \Rightarrow \boxed{\Delta\tau = \frac{4\pi}{f'_{r_0}}}$$

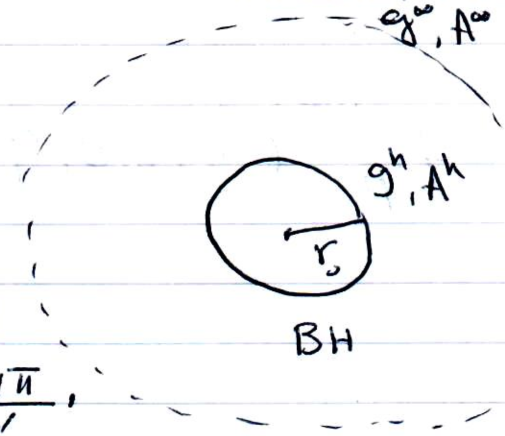
Thermodynamic Ensembles & Regularity Conditions

BH's thermodynamics is defined formally through Euclidean Path integrals, or

$$Z = \int D[g,A]\, e^{-I(g,A)},$$

where g and A are metric and matter field components.

$\ast$ As a boundary condition values of g, A must be fixed at the boundary.



$\ast$ We consider all possible values betw horizon and $r \to \infty$! But g, A should be finite at the two boundaries!

$\ast$ As we take $t \to i\tau$, we get a conical singularity at the horizon unless $[\tau] = \beta = \frac{4\pi}{f'(r_0)}$.

$\ast$ Norm of A, or $|A|^2 = A_\mu A^\mu$, must be finite at the horizon, e.g.

for R.N. soln $|A|^2 \sim -\frac{A_0^2}{f(r_0)} = \left(-\frac{q}{r_0} + c\right)^2 \cdot \frac{1}{f(r_0)}$ (at horizon!)

$\Rightarrow c = \frac{q}{r_0} = \phi_e$ (electric Pot.)

At $r=\infty$ $A_0 \to \phi_e$ (bndry Cond.) $\Longrightarrow$ $dM = TdS + \phi_e dQ_e$ $\to$ BH 1st law

$$Z \simeq \begin{cases} Tr\, e^{-\beta H} \simeq e^{-I_{on\text{-}shell}} = e^{-\beta F} & \to \text{Helm. free. Eng.} \\ Tr\, e^{-\beta(H-\mu_i C_i)} \simeq e^{-I_{on\text{-}shell}} = e^{-\beta G} & \to \text{Gibbs free. Eng.} \end{cases}$$

($\mu_i C_i$: $\phi_e Q_e$)

Work term $\phi_e dQ_e = e\phi_e dN = \mu dN$ (Ne)

Sch. soln $\quad I(T) = \beta F \quad$ (Cannonical Ens.)

R.N. soln $\quad I(T, \phi_e) = \beta G \quad$ (grand_can Ens.)

TN $\quad I(T, n) = \beta F \quad$ (Can. Ens.)

$$\overset{A_0}{A_\mu^{ele}} = \left(-\frac{q}{r} + C, 0, 0, 0\right) \xrightarrow{r \to \infty} (C, 0, 0, 0) \qquad A_\mu^{\infty} = C \ (\text{fixed}) + \text{finite}$$

$$\xrightarrow{r \to r_0} \left(-\cancelto{0}{\frac{q}{r_0}} + C, 0, 0, 0\right)$$

(fixed + finite)

$$A_\mu^{mag\text{-}ch.} \ (0, 0, 0, q_m(\cos\theta + C')) \xrightarrow{r \to \infty} A_\mu^{(m)\infty} = (0, 0, 0, q_m(\cos\theta\, C'))$$

$\downarrow$

(must be fixed)

$\xrightarrow{r \to r_0}$ is already regular.

$$\underset{\text{dives}}{\xrightarrow{\theta \to 0, \pi}} |A|^2 = q_m \frac{(\cos\theta + C')^2}{r^2 \sin^2\theta}$$

$$\Rightarrow A_\varphi^{(\pm)} = q_m(\cos\theta \pm 1)$$

TN Thermodynamics (our approach):

* Relaxing the cond. $\beta = 8\pi n$ and dealing with "n" as a conserved charge as well as considering $U = M - n\phi_n$ is the essence of our approach to construct a consistent thermo-dynamics of TN.

Caculated using counter-term regularization method

* $U = -\partial_\beta \ln Z = \partial_\beta I_{on\text{-}shell} = M - n\phi_n$.

$$dU = TdS + \phi_n dn + \phi_e dQ_e^\infty + \phi_m dQ_e^h.$$

* But the first law is not enough as a consistency check to our approach; but we have also two other important relations

* Gibb's-Duhem relation

$$\frac{I_{on\text{-}shell}}{\beta} = G = M - n\phi_n - TS - Q_e\phi_e$$

* Smarr's Relation

$$U = M - n\phi_n = 2TS + n\phi_e + Q_e\phi_e + Q_m\phi_m$$

(non-trivial !)

* 1st Law

$$dU = TdS + \phi_n dn + \phi_e dQ_e^\infty + \phi_m dQ_m^h.$$