The Standard Model of Particles and Interactions II- Towards Gauge Theories

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Some textbooks

Introductory textbooks:

-Introduction to High Energy Physics, 4th edition, D. Perkins (Cambridge)

-Introduction to Elementary particles, 2nd edition, D.Griffiths (Wiley)

Introduction to Quantum Field Theory:

-A Modern Introduction to Quantum Field Theory, Michele Maggiore (Oxford series)

-An Introduction to Quantum Field Theory, Peskin and Schroder (Addison Wesley)

-Quantum Field Theory, F. Mandl and G. Shaw, (Jhon Wiley & Sons)

Symmetries

I- Continuous global space-time (Poincaré) symmetries — all particles have (m, s)

-> energy, momentum, angular momentum conserved

II- Global (continuous) internal symmetries — -> B, L conserved (accidental symmetries)

III- Local or gauge internal symmetries — -> color, electric charge conserved

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

IV- Discrete symmetries — -> CPT

Why Quantum Field Theory (QFT)

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta-V\right)\Phi=0$$ Schrodinger equation

$$E=\frac{p^2}{2m}+V$$

$$E\rightarrow i\hbar\frac{\partial}{\partial t} \qquad p\rightarrow -i\hbar\frac{\partial}{\partial x}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\frac{m^2c^2}{\hbar^2}\right)\Phi=0$$ Klein Gordon equation

$$\left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\Psi=0$$ Dirac equation

Wave equations, relativistic or not, cannot account for processes in which the number and type of particles change.

We need to change viewpoint, from wave equation where one quantizes a single particle in an external classical potential to QFT where one identifies the particles with the modes of a field and quantize the field itself (second quantization).

Classical Field Theory

classical mechanics & lagrangian formalism

a system is described by $S = \int dt \mathcal{L}(q, \dot{q})$

position (q) momentum ($\dot{q}$)

action principle determines classical trajectory: $\delta S = 0$ --> Euler-Lagrange equations $\boxed{\dfrac{\partial \mathcal{L}}{\partial q_i} - \dfrac{\partial}{\partial t}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$

conjugate momenta $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$ hamiltonian $H(p, q) = \sum_i p_i \dot{q}_i - \mathcal{L}$

extend lagrangian formalism to dynamics of fields

$$S = \int d^4x \mathcal{L}(\varphi, \partial_\mu \varphi)$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

$\delta S = 0$ --> $\boxed{\dfrac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = 0}$

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

conjugate momenta $\Pi_i = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)}$ hamiltonian $H(x) = \sum_i \Pi_i(x) \partial_0 \varphi_i(x) - \mathcal{L}$

Classical Field theory and Noether theorem

Invariance of action under continuous global transformation ---> There is a conserved current/charge

$$\partial_\mu j^\mu = 0 \qquad Q = \int d^3x j^0(x,t)$$

example of transformation: $\boxed{\varphi \to \varphi e^{i\alpha}}$ (*)

if small increment $\alpha \ll 1$ $\varphi \to \varphi + i\alpha\varphi$

$$\delta\varphi = i\alpha\varphi$$

invariance of $\mathcal{L}$ under (*): $\delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$

Euler-Lagrange equations: $\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi$$

Scalar Field theory

Lorentz invariant action of a complex scalar field

$$S = \int d^4x (\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi)$$

Euler-Lagrange equation leads to Klein-Gordon equation

$$(\Box + m^2)\varphi = 0$$

with solution a superposition of plane waves:

$$\varphi(x) = \int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} (a_p e^{-ipx} + b_p^* e^{ipx})$$

existence of a global U(1) symmetry of the action

$$\varphi(x) \to e^{i\theta}\varphi(x)$$

conserved U(1) charge $Q_{U(1)} = \int d^3x j_0 \qquad j_\mu = i\varphi^* \overleftrightarrow{\partial}_\mu \varphi$

From first to second quantization

Basic Principle of Quantum Mechanics:

To quantize a classical system with coordinates q^i and momenta p^i, we promote q^i and p^i to operators and we impose $[q^i, p^j] = \delta^{ij}$

same principle can be applied to scalar field theory

where $q^i(t)$ are replaced by $\varphi(t,x)$
and $p^i(t)$ are replaced by $\Pi(t,x)$

φ and Π are promoted to operators and we impose $[\varphi(t,x), \Pi(t,y)] = i\delta^3(x-y)$

Expand the complex field in plane waves:

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_p e^{-ipx} + b_p^\dagger e^{ipx})$$

where a_p and b_p^+ are promoted to operators

$$[a_p, a_q^\dagger] = (2\pi^3)\delta^{(3)}(p-q) = [b_p, b_q^\dagger]$$

scalar field theory is a collection of harmonic oscillators

destruction operator $\boxed{a_p|0>= 0}$ defines the vacuum state |0>

a generic state is obtained by acting on the vacuum with the creation operators $|p_1 \dots p_n> \equiv a^\dagger_{p_1} \dots a^\dagger_{p_n}|0>$

Scalar field quantization continued

$$\mathcal{H} = \Pi\partial_0\varphi - \mathcal{L} \quad , \qquad \mathrm{H} = \int \frac{d^3p}{(2\pi)^3}\frac{E_p}{2}(a_p^\dagger a_p + b_p^\dagger b_p)$$

the quanta of a complex scalar field are given by two different particle species with same mass created by a^+ and b^+ respectively

The Klein Gordon action has a conserved U(1) charge due to invariance $\varphi(x) \to e^{i\theta}\varphi(x)$

$$Q_{U(1)} = \int d^3x j^0 = \int \frac{d^3p}{(2\pi)^3}(a_p^\dagger a_p - b_p^\dagger b_p)$$

2 different kinds of quanta: each particle has its antiparticle which has the same mass but **opposite U(1) charge**

Field quantization provides a proper interpretation of "E<0 solutions"

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_p e^{-ipx} + b_p^\dagger e^{ipx})$$

coefficient of the positive energy solution e^{-ipx} becomes after quantization the destruction operator of a particle while the coefficient of the e^{ipx} becomes the creation operator of its antiparticle

$a_p^+|0\rangle$ and $b_p^+|0\rangle$ represent particles with opposite charges

Similarly, we are led to quantize:

Spinor fields Ψ

Lorentz invariant lagrangian $\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu$

Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$

fermions: $\longrightarrow$ anticommutation relations $\{\Psi_a(x,t), \Psi_b^\dagger(y,t)\} = \delta^{(3)}(x-y)\delta_{ab}$

The electromagnetic field A_μ.

Lorentz inv. lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Maxwell eq. $\partial_\mu F^{\mu\nu} = 0$

Maxwell lagrangian inv. under $A_\mu \rightarrow A_\mu + \partial_\mu \theta$ Gauge transformation

The quantization of electromagnetic field is more subtle due to gauge invariance

Summary of procedure for building QFT

- Kinetic term of actions are derived from requirement of Poincaré invariance
- Promote field & its conjugate to operators and impose (anti) commutation relation
- Expanding field in plane waves, coefficients a_p, a_p^+ become operators
- The space of states describes multiparticle states

a_p destroys a particle with momentum p while a_p^+ creates it

e.g $$|p_1 \ldots p_n> \equiv a^\dagger_{p_1} \ldots a^\dagger_{p_n} |0>$$

$\longrightarrow$ crucial aspect of QFT: transition amplitudes between different states describe processes in which the number and type of particles changes

Gauge transformation and the Dirac action

Consider the transformation $\Psi \to \bar{e}^{iq\theta}\Psi$ U(1) transformation

it is a symmetry of the free Dirac action
if θ is constant

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

no longer a symmetry if $\theta = \theta(x)$

However, the following action is invariant under

$$\begin{cases} \Psi \to \bar{e}^{iq\theta}\Psi \\ A_\mu \to A_\mu + \partial_\mu \theta \end{cases}$$

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

where $\boxed{D_\mu \Psi = (\partial_\mu + iqA_\mu)\Psi}$ covariant derivative

We have gauged a global U(1) symmetry,
promoting it to a local symmetry

The result is a gauge theory and
A_μ is the gauge field

conserved current: $j^\mu = \bar{\Psi}\gamma^\mu\Psi$

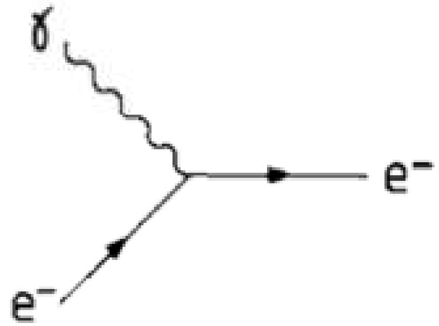
conserved charge: $Q = \int d^3x \bar{\Psi}\gamma^0\Psi = \int d^3x \Psi^\dagger \Psi$ $\rightsquigarrow$ electric charge

Electrodynamics of a spinor field

$$\boxed{\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi}$$ where $$\boxed{D_\mu \Psi = (\partial_\mu + iqA_\mu)\Psi}$$

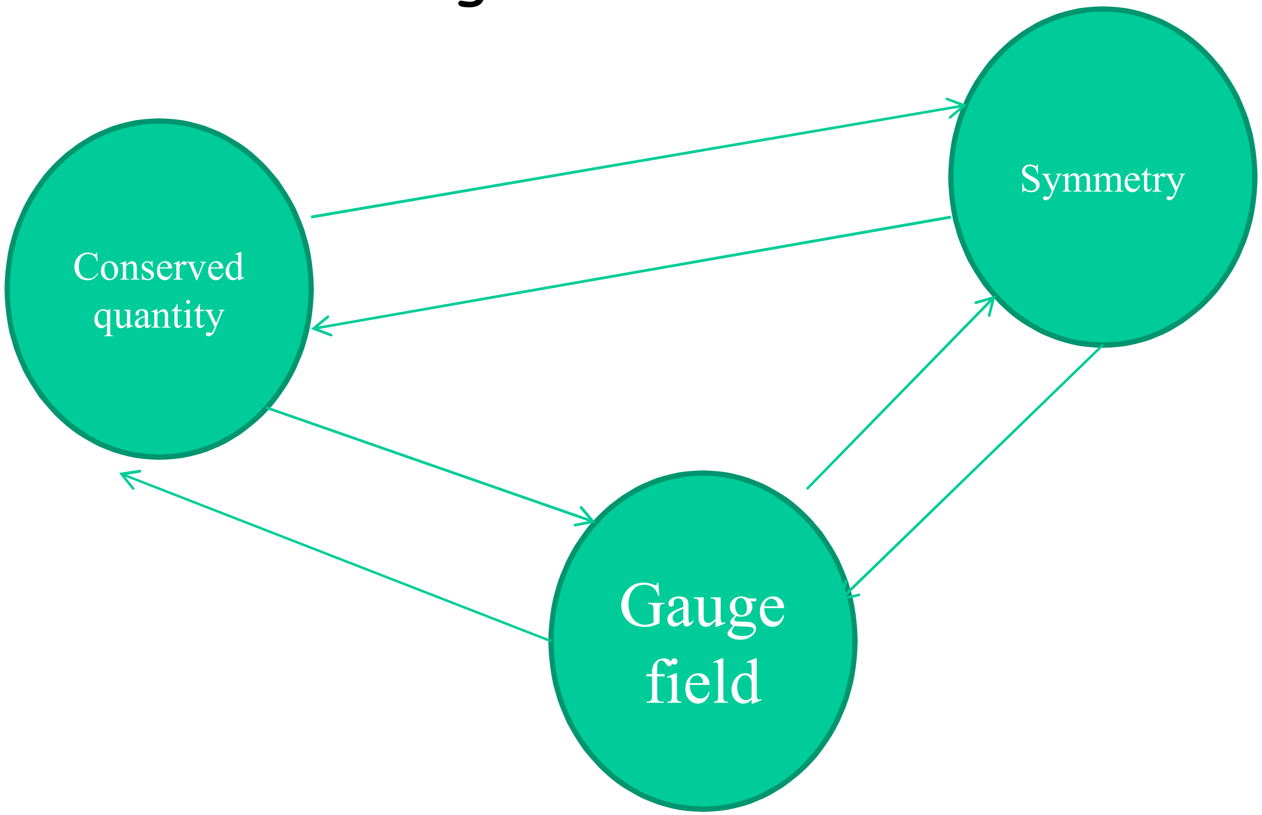
$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - qA_\mu \bar{\Psi}\gamma^\mu \Psi$$

Coupling of the gauge field A_μ to the current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$



Gauge Symmetry predicts dynamics

1. The photon is massless
2. The minimal coupling
3. There is no self coupling for photon
4. Conservation of charge



Yang-Mills fields

These transformations are elements of U(1) group

$$\Psi \rightarrow e^{-iq\theta}\Psi$$

In the electroweak theory , more complicated transformations, belonging to the SU(2) group are involved

$$\Psi \rightarrow \exp(-ig\ \tau.\lambda)\Psi$$

where $\tau = (\tau_1, \tau_2, \tau_3)$ are three 2*2 matrices

Generalization to SU(N)

N^2-1 generators (N×N matrices)

$$\Psi(x) \rightarrow U(x)\Psi(x)$$

$$U(x) = e^{-ig\theta^a(x)T^a}$$

$$A_\mu(x) \rightarrow U A_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger$$

Gauge theories: Electromagnetism (EM) & Yang-Mills

EM U(1) $\phi \rightarrow e^{-i\alpha}\phi$ but $\partial_\mu\phi \rightarrow e^{-i\alpha}(\partial_\mu\phi) - \underbrace{i(\partial_\mu\alpha)}_{\neq 0 \text{ if local transformations}}\phi\, e^{-i\alpha}$

EM field and covariant derivative $\partial_\mu\phi + ieA_\mu\phi \rightarrow e^{i\alpha}(\partial_\mu\phi + ieA_\mu\phi)$

if $A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$

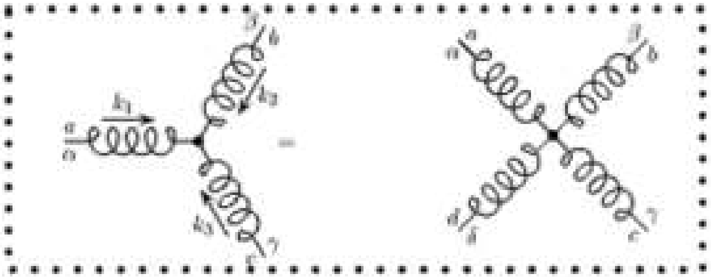
the EM field keep track of the phase in different points of the space-time

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Yang-Mills : non-abelian transformations $\phi \rightarrow U\phi$

$\partial_\mu\phi + igA_\mu\phi \rightarrow U(\partial_\mu\phi + igA_\mu\phi)$ if $A_\mu \rightarrow UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1}$

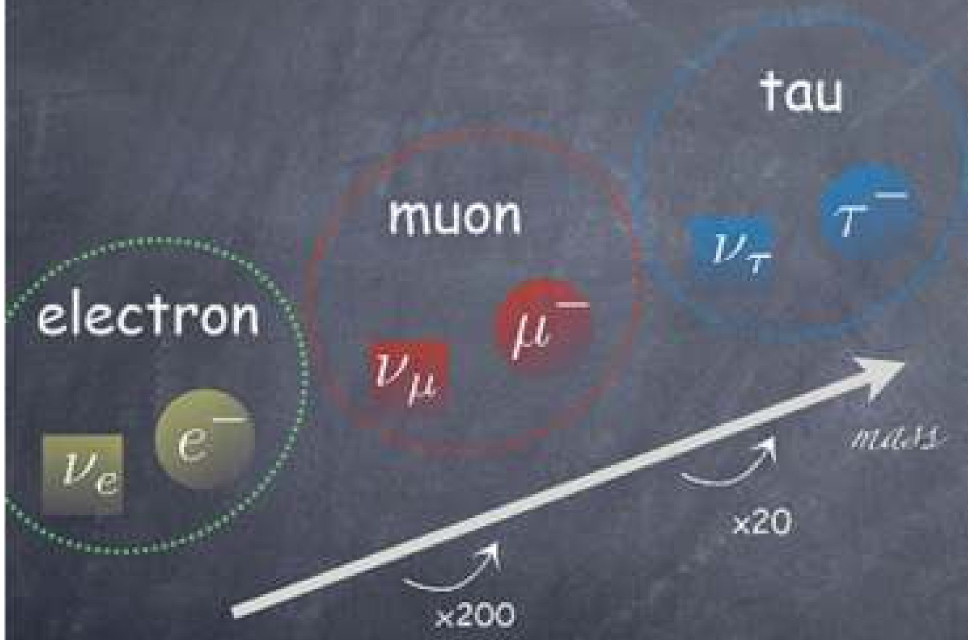
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{ig[A_\mu, A_\nu]}_{\text{non-abelian int.}}$$



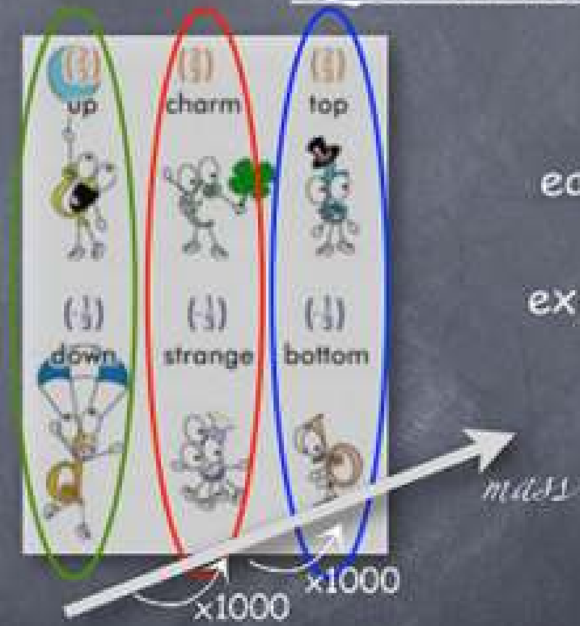
The Standard Model: matter

the elementary blocks:

<u>LEPTONS</u>



<u>QUARKS</u>



each of the 6 quarks exists in three colors

<u>composite states</u> (white objects

- baryons
  - proton $p = (u, u, d)$
  - neutron $n = (u, d, d)$
- mesons

+ antiparticles

The Standard Model : interactions

- $U(1)_Y$ — *electromagnetic interactions*

  Photon γ

  light, atoms, molecules — 10^{-6}

- $SU(2)_L$ — *weak interactions*

  bosons $W^{\pm}, Z^0$

  β decay

  $$n \xrightarrow{W^{\pm}} p + e^- + \bar{\nu}_e$$

  10^{-2}

- $SU(3)_C$ — *strong interactions*

  gluons g^a

  atomic nuclei

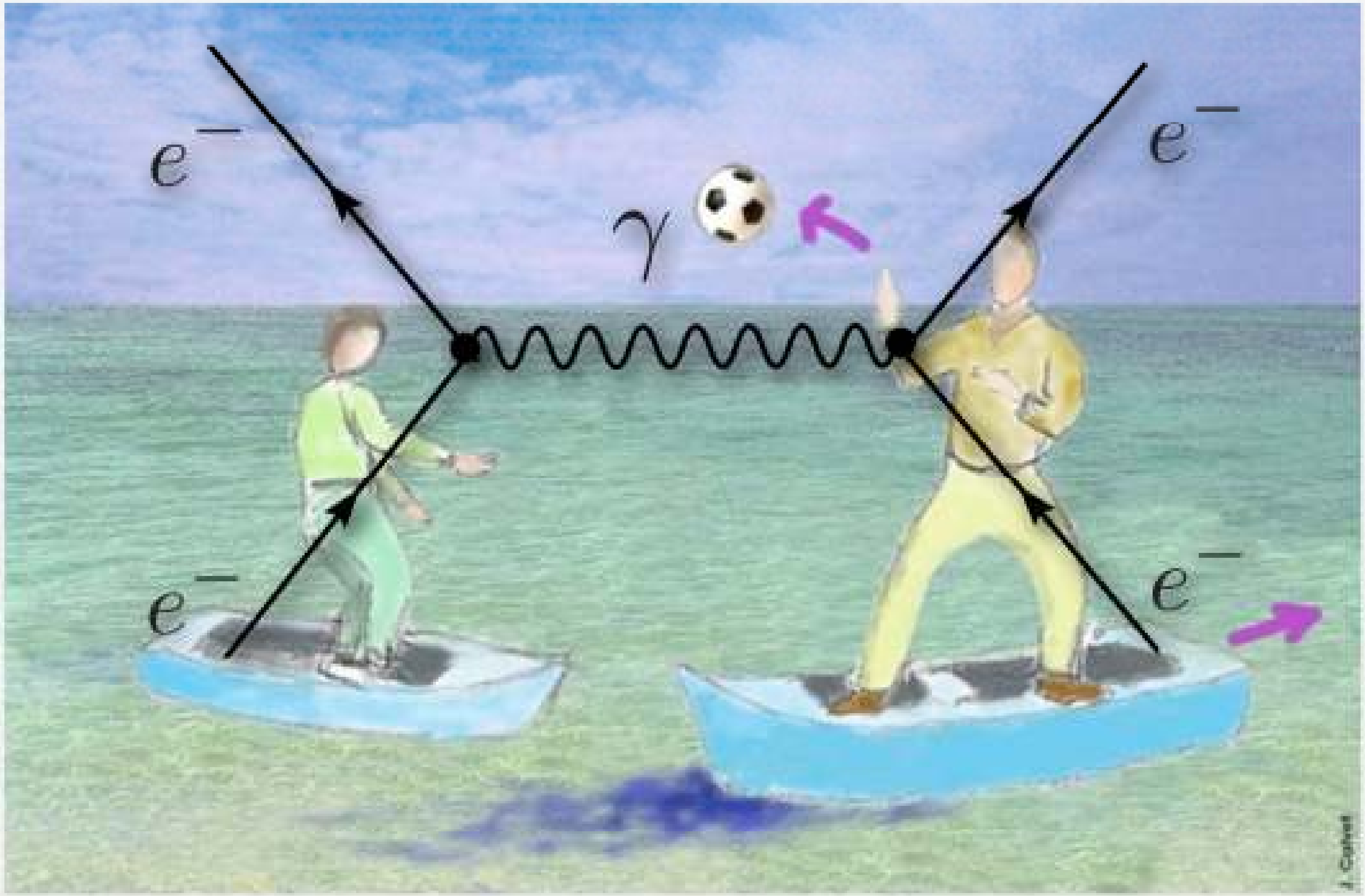
  α decay

  $$^{238}_{92}U \rightarrow {}^{234}_{90}Th + {}^{4}_{2}He$$

  1

strength

Interactions between particles



Elementary particles interact with each other by exchanging gauge bosons

The beauty of the SM comes from the the identification of a unique dynamical principle describing interactions that seem so different from each others

$\boxed{\text{gauge theory = spin-1}}$

The Lagrangian of the world

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \\
& +\bar{Q}_i i\not{D}Q_i + \bar{u}_i i\not{D}u_i + \bar{d}_i i\not{D}d_i + \bar{L}_i i\not{D}L_i + \bar{e}_i i\not{D}e_i \\
& +Y_u^{ij}\bar{Q}_i u_j\tilde{H} + Y_d^{ij}\bar{Q}_i d_j H + Y_l^{ij}\bar{L}_i e_j H + |D_\mu H|^2 \\
& -\lambda(H^\dagger H)^2 + \lambda v^2 H^\dagger H + \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}
\end{aligned}
$$

What about baryon and lepton numbers? -> accidental symmetries!