Stars and Black Holes in General Relativity

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Limitations of the Schwarzschild Metric

- The Schwarzschild solution describes a static, spherically symmetric, **vacuum** spacetime.
- It predicts black holes, white holes, and wormholes in its maximal extension.
- **Reality Check:** Such features are unlikely in the real world.
  - Real astrophysical objects contain matter ($T_{\mu\nu} \neq 0$).
  - Birkhoff's theorem states that any spherically symmetric vacuum region is described by part of the Schwarzschild metric.
  - However, the presence of matter dramatically alters the global spacetime picture.

From Stars to Black Holes

- A static spherical object (star) with radius $R > 2GM$ has a Schwarzschild exterior.
- No singularities or horizons exist *within* the star.
- **Stellar Evolution:** Stars evolve and may collapse under gravity.
- If a star shrinks below $r = 2GM$, it forms a black hole with a singularity and an event horizon.
- **No White Holes/Wormholes:** Realistic stellar collapse models (e.g., conformal diagrams) show a future event horizon and singularity, but no white hole, past horizon, or separate asymptotic regions.
- This chapter focuses on understanding static configurations describing the interiors of spherically symmetric stars.

Metric for Non-Vacuum Solutions

- For a general static, spherically symmetric spacetime (now with matter), the metric is:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

- Our goal is to find the functions $\alpha(r)$ and $\beta(r)$ by solving the full Einstein equations.
- We now use the full Einstein Field Equation ($G_{\mu\nu} = 8\pi G T_{\mu\nu}$), as we are looking for non-vacuum solutions.

Full Einstein Equations and Matter Model

- **Einstein Tensor Components ($G_{\mu\nu}$):**
  - $G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)}(2r\partial_r\beta - 1 + e^{2\beta})$
  - $G_{rr} = \frac{1}{r^2}(2r\partial_r\alpha + 1 - e^{2\beta})$
  - $G_{\theta\theta} = r^2 e^{-2\beta}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)]$
  - $G_{\phi\phi} = \sin^2\theta\ G_{\theta\theta}$
- **Matter Model (Perfect Fluid):** We model the star as a perfect fluid with energy density $\rho(r)$ and pressure $p(r)$.
  - Stress-Energy Tensor: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$
  - For static solutions, the 4-velocity is $U_\mu = (e^\alpha, 0, 0, 0)$ (normalized to $U^\mu U_\mu = -1$).
  - Components of $T_{\mu\nu}$: $T_{tt} = e^{2\alpha}\rho,\ T_{rr} = e^{2\beta}p,\ T_{\theta\theta} = r^2 p,$ $T_{\phi\phi} = r^2\sin^2\theta p.$

Three Independent Equations

Equating the components of $G_{\mu\nu}$ and $8\pi G T_{\mu\nu}$ yields three independent equations:

1. **tt-component:**

$$\frac{1}{r^2} e^{-2\beta}(2r\partial_r\beta - 1 + e^{2\beta}) = 8\pi\ G\rho$$

2. **rr-component:**

$$\frac{1}{r^2} e^{-2\beta}(2r\partial_r\alpha + 1 - e^{2\beta}) = 8\pi\ Gp$$

3. **$\theta\theta$-component:**

$$e^{-2\beta}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)] = 8\pi\ Gp$$

Mass Function and Metric Generalization

- To simplify the equations, we introduce a new function $m(r)$:

$$m(r) = \frac{1}{2G}(r - re^{-2\beta}) \quad \text{or equivalently} \quad e^{2\beta} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}$$

- The metric then becomes:

$$ds^2 = -e^{2\alpha(r)}dt^2 + \left[1 - \frac{2Gm(r)}{r}\right]^{-1} dr^2 + r^2 d\Omega^2$$

- This form for g_{rr} is a direct generalization of the Schwarzschild metric.

Mass within a Radius

- The tt-component of Einstein's equation (with $m(r)$) integrates to:

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad \Longrightarrow \quad m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

- This $m(r)$ is interpreted as the mass (or energy) contained within radius r.
- At the star's surface ($r = R$), $m(R)$ must match the Schwarzschild mass M of the exterior solution:

$$M = m(R) = 4\pi \int_0^R \rho(r) r^2 dr$$

- **Binding Energy:** The true integrated energy density $\overline{M} = 4\pi \int_0^R \rho(r) r^2 e^{\beta(r)} dr$ is generally greater than M. The difference $E_B = \overline{M} - M > 0$ represents the gravitational binding energy.

Pressure Gradient in a Star

- The rr-component of Einstein's equation can be written as:

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi G r^3 p}{r[r - 2Gm(r)]}$$

- Instead of using the $\theta\theta$ equation directly, we use the energy-momentum conservation equation ($\nabla_\mu T^{\mu\nu} = 0$).
- The r-component of $\nabla_\mu T^{\mu\nu} = 0$ gives:

$$(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}$$

- Combining these two equations (eliminating $\alpha(r)$) yields the **Tolman-Oppenheimer-Volkoff (TOV) Equation**:

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]}$$

- This is the equation of hydrostatic equilibrium for a spherically symmetric star in GR.

Closing the System of Equations

- The TOV equation relates $p(r)$ to $\rho(r)$ and $m(r)$.
- To get a closed system, we need an **equation of state** ($p = p(\rho)$), which relates pressure to energy density.
- Common astrophysical equations of state include polytropes ($p = K\rho^\gamma$).

A Simple Model: Constant Density Star

- A simple model assumes an incompressible fluid: density $\rho(r) = \rho_*$ (constant) for $r < R$, and 0 for $r > R$.
- Integrating $dm/dr = 4\pi r^2 \rho$:

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_*, & r < R \\ \frac{4}{3}\pi R^3 \rho_* = M, & r > R. \end{cases}$$

- Integrating the TOV equation with this $\rho(r)$ gives $p(r)$. The pressure increases towards the core.
- The metric component $e^{2\alpha(r)}$ can also be found.

A Simple Model: Constant Density Star

- The next step is to obtain the pressure through integrating hydrostatic equilibrium equation;

$$p(r) = \rho_* \left[\frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R-2GM}} \right].$$

- Now integrate the *rr*-component of Einstein's equation to obtain $\alpha(r)$.

$$e^{\alpha(r)} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2}, \quad r < R.$$

Limits to Stellar Stability

- For a star of fixed radius R, the central pressure $p(0)$ becomes infinite if its mass exceeds a maximum value:

$$M_{\text{max}} = \frac{4}{9G}R$$

- This implies that if we try to squeeze more mass than M_{max} into a radius R, General Relativity admits no static solutions.
- Such a star must inevitably continue shrinking, eventually forming a black hole.
- **Buchdahl's Theorem:** A more general result states that for any reasonable static, spherically symmetric interior solution, $M < \frac{4}{9G}R$. This reinforces the idea that there's a limit to how compact a star can be without collapsing.

Stars Supported by Electron Degeneracy Pressure

- After nuclear fuel is exhausted, stars shrink under gravity.
- Collapse may be halted by **Fermi degeneracy pressure** from electrons (Pauli exclusion principle).
- A stellar remnant supported by electron degeneracy pressure is called a **white dwarf**.
- Typical white dwarfs are Earth-sized and are the end state for most stars.

Beyond the Chandrasekhar Limit

- If a star's mass exceeds the **Chandrasekhar limit** (approx. $1.4M_\odot$), electron degeneracy pressure is insufficient.
- Electrons combine with protons to form neutrons and neutrinos (inverse beta decay).
- The result is a **neutron star**, with a typical radius of about 10 km.
- Neutron stars often manifest as **pulsars** (rapidly spinning, highly magnetized).

Beyond the Oppenheimer-Volkoff Limit

- If a neutron star's mass exceeds the **Oppenheimer-Volkoff limit** (current estimates: 3-4 $M_\odot$), even neutron degeneracy pressure cannot resist gravity.
- The star is believed to continue collapsing, forming a **black hole**.
- **Detection:** Black holes don't emit light (neglecting Hawking radiation).
  - Around BH's accretion disks form, heating up and emitting X-rays.
  - The Event Horizon Telescope has captured the first images of supermassive black holes at the center of M87 and the Milky Way.
- **Classes of Black Holes:**
  - **Stellar-mass black holes:** $\sim$ solar mass, endpoints of massive stars.
  - **Supermassive black holes:** $10^6 - 10^9 M_\odot$, found at galactic centers (e.g., Sgr A* in Milky Way).
  - Other possibilities: primordial, middleweight black holes.
- Most observed black holes are expected to be spinning due to accretion disk dynamics.