Introduction to General Relativity

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Special Relativity: The Foundation

- **Core Principles:**
  1. **Constant Speed of Light:** The speed of light in a vacuum (c) is constant in all inertial reference frames (IRF).
  2. **Relativity Principle:** The laws of physics have the same form in all inertial reference frames.
- **Implications:** Space and time are not absolute but are intertwined into a single entity called spacetime. Time is relative, space is relative and mass is energy.

Special Relativity: Lorentz Transformations

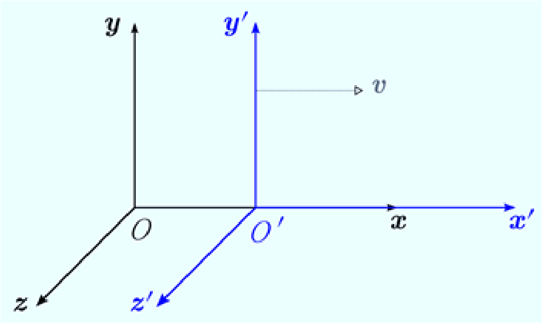


Figure: Two inertial frames: Σ' is moving relative to Σ

Lorentz Transformations

Consider two IRFs, Σ and Σ', where Σ' moves with uniform velocity $\vec{v} = v\hat{x}$ relative to Σ. The relations between coordinates are

$$\begin{aligned} ct' &= \gamma(ct - \beta\, x) \\ x' &= \gamma(x - \beta\, ct) \\ y' &= y \\ z' &= z \end{aligned}$$

Where $\beta = \frac{v}{c}$ and $\gamma = (1 - \beta^2)^{-1/2}$ (Lorentz factor).

Matrix Form ($x'^\mu = \Lambda^\mu{}_\nu x^\nu$):

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Here, $x^\mu \equiv (ct, x, y, z)$ is a 4-vector (using ct for time component for consistent units).

The Spacetime Interval

A Quantity Unchanged by Lorentz Transformations

Lorentz transformations leave the following quantity invariant:

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2 = S^2$$

The nature of this interval classifies event relationships:

- $S^2 < 0$: **Time-like interval**.
  - Temporal separation $>$ spatial separation.
  - Events are causally connected ($v < c$). An object can travel between them.
- $S^2 = 0$: **Null-like interval**.
  - Events are connected by a light ray ($v = c$).
- $S^2 > 0$: **Space-like interval**.
  - Spatial separation $>$ temporal separation.
  - Events are not causally connected ($v > c$ would be required).

Special Relativity: Light Cone

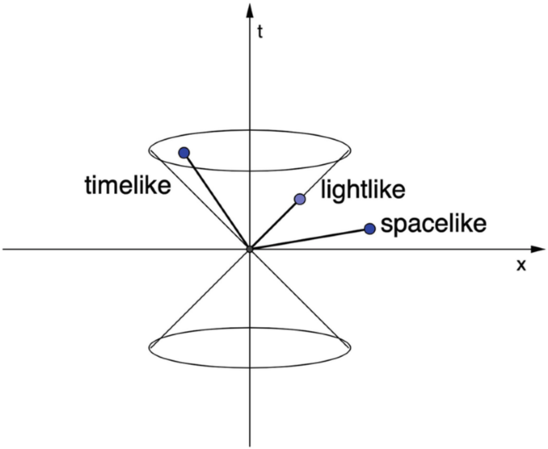


Figure: timelike, spacelike and lightlike intervals

Spacetime Four-Vector

- **Concept:** In Special Relativity, space and time are unified into a single four-dimensional entity called **spacetime**. Events are points in this spacetime.
- **Position Four-Vector (x^μ):** Combines the time coordinate (multiplied by c) and the three spatial coordinates into a single vector.
  - $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$
  - Where μ is an index ranging from 0 to 3.
- **Lorentz Invariance:** The "length" or "interval" between two events in spacetime is invariant (the same for all inertial observers), even though individual space and time components change.
  - $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ (Minkowski metric)
- **Differential Operator:** $\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

  Also, $\partial_\mu \partial^\mu = -{\partial_0}^2 + {\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2 = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Four-Momentum

- **Concept:** Just as space and time are unified, energy and momentum are also combined into a single four-vector in Special Relativity.
- **Momentum Four-Vector (p^μ):** Combines the relativistic energy (divided by c) and the three spatial momentum components.
  - $p^\mu = (p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$
  - Where $E = \gamma mc^2$ is the relativistic energy and $\mathbf{p} = \gamma m\mathbf{v}$ is the relativistic momentum.
- **Lorentz Invariance:** The "length squared" of the four-momentum vector is also invariant and related to the rest mass of the particle:
  - $-(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -(E/c)^2 + |\mathbf{p}|^2 = -(mc)^2$
  - This leads directly to the famous mass-energy equivalence $E^2 = (pc)^2 + (mc^2)^2$.
- **Significance:** Writing down laws of physics using 4-vectors or scalar ensures that the physics is covariant under Lorentz transformations.

Maxwell's Equations and Relativity

- **The Challenge:** Maxwell's equations were inherently consistent with the second postulate of special relativity (constant speed of light). However, their form was not obviously invariant under Galilean transformations, implying a preferred reference frame (the aether).
- **Einstein's Insight:** Special Relativity showed that Maxwell's equations are *already* relativistically correct. The problem was with Newtonian mechanics, not with electromagnetism.
- **The Unification:** Special Relativity provided the framework to show that electric and magnetic fields are not independent entities but are two aspects of a single **electromagnetic field**, whose appearance depends on the observer's frame of reference.

Introducing the Electromagnetic Field Tensor

- Need for Covariant Form: To express Maxwell's equations in a way that is explicitly invariant under Lorentz transformations, we use **tensor notation**. This makes the relativistic nature of the equations transparent.
- We introduce a four-potential $A_\mu = (\phi/c, \mathbf{A})$ and a 2nd rank anti-symmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
- Electromagnetic Field Tensor ($F^{\mu\nu}$): It combines the electric field ($\mathbf{E}$) and magnetic field ($\mathbf{B}$) into a single mathematical object.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Maxwell's Equations in Covariant Form

- **Two Tensor Equations:** The four original Maxwell's equations can be elegantly expressed as two tensor equations:
  1. **First Pair (Homogeneous Equations):** Describes the absence of magnetic monopoles and Faraday's law of induction.

     $$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

  2. **Second Pair (Inhomogeneous Equations):** Describes Gauss's law and Ampere-Maxwell's law.

     $$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

     - J^ν: Four-current density vector $(c\rho, \mathbf{J})$.
     - μ_0: Permeability of free space.
- **Significance:** These covariant forms explicitly demonstrate the Lorentz invariance of Maxwell's equations, confirming their consistency with Special Relativity.

Inertial vs. Gravitational Mass

- **Inertial Mass (m_I):** Defined by Newton's $F = m_I a$. Resistance to acceleration.
- **Gravitational Mass (m_G):** Defined by Newton's $F_G = m_G \frac{M_G G_N}{r^2}$. Source of, or response to, gravitational field.
- **Empirical Fact:** $m_I = m_G$. This leads to gravitational acceleration $a_G = \frac{G_N M_G}{r^2}$ being independent of the object's mass or composition.
- **Key Insight:** Gravity is "blind" to the physical content of any object. This suggests gravity might be a manifestation of spacetime geometry rather than a force since all objects are affected in the same way.

Gravity as a spacetime curvature

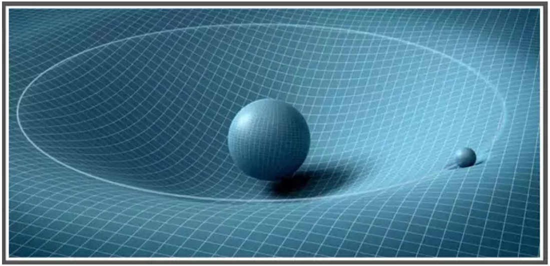
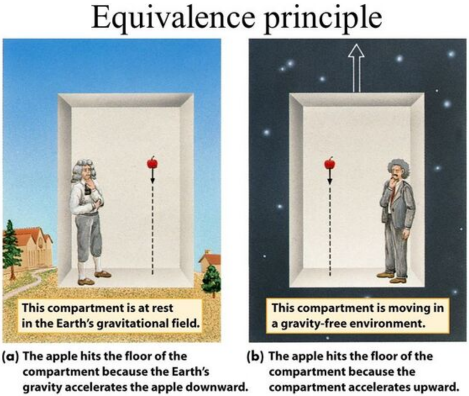


Figure: Gravitational acceleration does not depend on the mass of the falling object!

Einstein's Elevator Experiment

- An observer in a windowless elevator cannot distinguish between:
  - Being in a uniformly accelerating rocket in deep space.
  - Being at rest in a uniform gravitational field.
- Similarly, free-falling in a gravitational field is locally indistinguishable from being in deep space (weightlessness).
- **Conclusion (Equivalence Principle):** Uniform acceleration is locally equivalent to a uniform gravitational field.
- **Consequence:** Gravity affects the very fabric of spacetime, causing it to bend or curve. This curvature dictates the paths objects take, replacing the classical "force" of gravity.

Equivalence Principle



Special Relativity: Lorentz Transformations

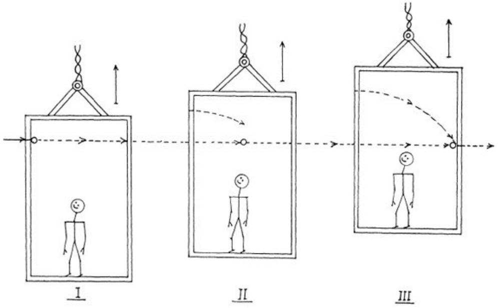


Figure: If uniform acceleration affects light gravity must affect it in the same way!

Why Tensors?

- General Relativity describes gravity in **general (arbitrary) reference frames**, not just inertial ones (like SR).
- We need mathematical quantities whose transformation laws are well-defined under general coordinate transformations: $x^\mu \equiv (x^0, x^1, x^2, x^3) \longrightarrow x'^\mu \equiv (x'^0, x'^1, x'^2, x'^3)$.
- Unlike SR's linear Lorentz transformations ($A'^\mu = \Lambda^\mu{}_\nu A^\nu$), GR uses non-linear transformations where derivatives like $\frac{\partial x'^\mu}{\partial x^\nu}$ become crucial.

Definition: A tensor is a geometric object that transforms in a well-defined way under coordinate transformations.

Types of Tensors

1. Rank (0,0) Tensor (Scalar): A quantity invariant under coordinate transformations.

$$\phi'(x') = \phi(x)$$

   Examples: Spacetime interval, mass, temperature.

2. Rank (1,0) Tensor (Contravariant Vector): Transforms like coordinates (dx^μ).

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

   Examples: 4-position, 4-velocity.

3. Rank (0,1) Tensor (Covariant Vector): Transforms like gradients ($\frac{\partial}{\partial x^\mu}$).

$$A'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} A_\mu$$

   Examples: Gradient of a scalar field, 4-momentum with lower index.

Types of Tensors

General Rank (k, l) **Tensor:** k upper (contravariant) indices, l lower (covariant) indices.

$$T'^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial x'^{\nu_l}} \right) T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l}$$

Properties of Tensors

- **Contraction:** Summing over one contravariant and one covariant index of a tensor reduces its rank by 2. Example: Contraction of A'^{μ} and B'_{μ} forms a scalar:

$$A'_{\mu}B'^{\mu} = \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}A_{\alpha}\right)\left(\frac{\partial x'^{\mu}}{\partial x^{\beta}}B^{\beta}\right) = \frac{\partial x^{\alpha}}{\partial x^{\beta}}A_{\alpha}B^{\beta} = \delta^{\alpha}_{\beta}A_{\alpha}B^{\beta} = A_{\alpha}B^{\alpha}$$

- **Symmetric and Anti-symmetric Tensors:**
  - Symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$
  - Anti-symmetric: $T^{\alpha\beta} = -T^{\beta\alpha}$
- **Decomposition:** Any tensor $B_{\mu\nu}$ can be uniquely decomposed into symmetric and anti-symmetric parts:

$$B_{\mu\nu} = B_{(\mu\nu)} + B_{[\mu\nu]}$$

where $B_{(\mu\nu)} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$ and $B_{[\mu\nu]} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$.

The Fundamental Role of $g_{\mu\nu}$

- The **metric tensor** ($g_{\mu\nu}$) is central to General Relativity.
- It defines infinitesimal distances between any two points in curved spacetime. It is a generalization of the Pythagorean theorem.

The square of an infinitesimal spacetime interval dS^2:

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + 2g_{01} dx^0 dx^1 + g_{22}(dx^2)^2 + ...$$

In flat spacetime, $g_{\mu\nu}$ reduces to the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ (using $(-, +, +, +)$ signature).

Raising and Lowering Indices: The metric tensor and its inverse ($g^{\mu\nu}$) are used to transform between covariant and contravariant forms of tensors:

- Lowering: $A_\mu = g_{\mu\nu} A^\nu$
- Raising: $A^\mu = g^{\mu\nu} A_\nu$
- Example: $g_{\nu\alpha} T^{\mu\alpha} = T^\mu_\nu$

The Problem with Partial Derivatives

- Partial derivatives of scalars (ϕ) are tensors: $\frac{\partial \phi'}{\partial x'^{\mu}} = \frac{\partial \phi}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}$.
- However, the partial derivative of a vector $A^{\mu}(x)$ **does not transform as a tensor**:

$$\frac{\partial A'^{\mu}}{\partial x'^{\nu}} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial A^{\alpha}}{\partial x^{\beta}} + \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x^{\alpha}} A^{\alpha}$$

The second term, involving second partial derivatives of the coordinate transformation, prevents it from being a tensor. This highlights that basis vectors change in curved spacetime.

Covariant Derivative

Consider a vector $V = V^\mu \hat{e}_\mu$. Its total variation δV includes changes in components and changes in basis vectors:

$$\delta V = \delta V^\mu \hat{e}_\mu + V^\mu \delta \hat{e}_\mu$$

The change in basis vectors can be expressed using Christoffel symbols:

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \Gamma^\alpha_{\mu\nu} \hat{e}_\alpha$$

Substituting this into the derivative of V with respect to x^β:

$$D_\beta V = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\nu} V^\mu \frac{\partial x^\nu}{\partial x^\beta} \right) \hat{e}_\alpha = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right) \hat{e}_\alpha$$

The term in parentheses is precisely the definition of the covariant derivative $\nabla_\beta V^\alpha$. It accounts for the non-Euclidean nature of parallel transporting a vector in curved space.

Covariant Derivative

To define differentiation that yields a tensor in curved spacetime, we introduce the **covariant derivative**, ∇_ν.

- For a contravariant vector A^μ:

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\alpha\nu} A^\alpha$$

- For a covariant vector A_μ:

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\alpha$$

- For a mixed tensor $H^\mu{}_\nu$:

Christoffel Symbol ($\Gamma^\sigma_{\mu\nu}$): Not a tensor, but represents how basis vectors change in curved space. It is calculated from the metric tensor:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

A crucial property: $\nabla_\alpha g_{\mu\nu} = 0$ (metric is covariantly constant).

Generalizing Straight Lines to Curved Spacetime

- In Euclidean space, a straight line is the shortest path between two points, and its tangent vector does not change direction.
- A **geodesic** is the generalization of a straight line to curved spaces or spacetimes.
- It is the curve of extremal (minimum or maximum) length (interval) between two points.
- Equivalently, it's the path along which a particle moves when only acted upon by gravity (i.e., its 4-velocity is parallel transported along its path).

3D Curve Example: A curve $\overline{X}(s) = X(s)\hat{e}_x + Y(s)\hat{e}_y + Z(s)\hat{e}_z$ has a tangent vector $\overline{t} = \frac{d\overline{x}}{ds}$.

The Geodesic Equation

From Parallel Transport or Variational Principle

From Parallel Transport: For a geodesic, the covariant derivative of the tangent vector $t^\alpha = \frac{dx^\alpha}{ds}$ along the curve must be zero (or proportional to itself for a non-affine parameter).

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \lambda(s)\left(\frac{dx^\alpha}{ds}\right)$$

If we can find a parameter s' such that $\lambda(s') = 0$, it's called an **affine parameter** (e.g., proper time for massive particles). The equation becomes:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

This is the geodesic equation.

From Variational Principle (Euler-Lagrange): Consider a free particle with Lagrangian $L = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$. Applying the Euler-Lagrange equations $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$ yields the same geodesic equation:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

Relativistic Kinematics and Charge Conservation

- **4-Velocity (u^μ):** For a massive particle, in its rest frame, time is measured by proper time τ. The spacetime interval is $dS^2 = -c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

$$-c^2 = \eta_{\mu\nu} \left(\frac{dx^\mu}{d\tau} \right) \left(\frac{dx^\nu}{d\tau} \right)$$

  The 4-velocity is $u^\mu = \frac{dx^\mu}{d\tau}$. Its components are $u^\mu \equiv (\gamma c, \gamma v^i)$.
- **4-Current (J^μ):** Generalizes charge density and current to spacetime.
  - Classical continuity equation: $\frac{\partial \rho}{\partial t} + \overline{\nabla} \cdot \overline{J} = 0$.
  - Relativistic charge density: $\rho = \rho_0 \gamma$ (where ρ_0 is rest charge density).
  - 4-current: $J^\mu \equiv (\rho_0 \gamma c, \rho_0 \gamma \overline{u}) = \rho_0 u^\mu$ (using c in u^0 for proper units).
  - Covariant continuity equation: $\partial_\mu J^\mu = 0$. This implies charge conservation.

Energy and Momentum Distribution

- For matter particles with rest mass m_0, the rest mass density is $\rho_0 = m_0 n$.
- The distribution of energy and momentum in spacetime is described by the **Stress-Energy Tensor** (or Energy-Momentum Tensor), $T^{\mu\nu}$.
- For a simple collection of dust particles (no pressure):

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu$$

- **Components and Physical Interpretation:**
  - $T^{00} = \rho_0 (u^0)^2 = \rho_0 \gamma^2 c^2$: Energy/mass density.
  - $T^{0i} = \rho_0 u^0 u^i = \rho_0 \gamma c u^i$: Energy flux in i-direction (or mass current).
  - $T^{i0} = \rho_0 u^i u^0 = \rho_0 \gamma c u^i$: Momentum density in i-direction (or energy flux).
  - $T^{ij} = \rho_0 u^i u^j$: Momentum flux (stress or pressure).

Perfect Fluid Stress-Energy Tensor

- For a perfect fluid (isotropic, no viscosity, no heat conduction) with energy density ρ and isotropic pressure P:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + \eta^{\mu\nu} P \quad \text{(in SR)}$$

- In curved spacetime, the flat metric $\eta^{\mu\nu}$ is replaced by the spacetime metric $g^{\mu\nu}$:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + g^{\mu\nu} P \quad \text{(in GR)}$$

- **Conservation of Energy and Momentum:** In GR, this is expressed by the covariant divergence of the stress-energy tensor being zero: $\nabla_\mu T^{\mu\nu} = 0$.

Connecting to Classical Gravity

- **Classical Gravity (Newtonian):**
  - Poisson's equation (field equation): $\nabla^2\phi = 4\pi G\rho$.
  - Newton's second law (test particle equation): $\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i}$.
- In GR, Newton's second law for a test particle should emerge from the geodesic equation in the following limits:
  - **Weak field:** $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($h_{\mu\nu}$ is small).
  - **Non-relativistic:** Particle velocities $v \ll c$, so $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \simeq c$.
  - **Static:** $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$.

From Geodesics to Potential

The geodesic equation: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$. In the weak-field, non-relativistic, static limit, the dominant term is for $\alpha = \beta = 0$:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0$$

Since $\frac{dx^0}{d\tau} \approx c$, we have $\frac{d^2x^\mu}{d\tau^2} + c^2\Gamma^\mu_{00} = 0$.
Evaluating Γ^μ_{00}:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{0\alpha} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{\mu\alpha}\frac{\partial g_{00}}{\partial x^\alpha}$$

Using $g^{\mu\alpha} \approx \eta^{\mu\alpha}$ and $g_{00} = \eta_{00} + h_{00} = -1 + h_{00}$:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}\eta^{\mu\alpha}\frac{\partial h_{00}}{\partial x^\alpha}$$

For $\mu = i$ (spatial component):

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}$$

Comparing with Newton's second law ($\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i}$, with $d\tau \approx dt$):

From Poisson's Equation to a Tensor Form

Substitute g_{00} into Poisson's equation $\nabla^2\phi = 4\pi G\rho$:

$$\nabla^2\left(-\frac{c^2}{2}(g_{00}+1)\right) = 4\pi G\rho$$

$$-\frac{c^2}{2}\nabla^2 g_{00} = 4\pi G\rho = \frac{4\pi G}{c^2}T_{00}$$

$$\nabla^2 g_{00} = -\frac{8\pi G}{c^4}T_{00}$$

This suggests a generalization for a full covariant field equation:

$$\nabla^2 g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

However, this is **not correct** because $\nabla_\alpha g_{\mu\nu} = 0$ (covariant derivative of metric is zero). The left side would be zero, but $T_{\mu\nu}$ is generally non-zero.

Guiding Principle for EFE: We need an equation of the form $X_{\mu\nu} = KT_{\mu\nu}$ such that $\nabla^\mu X_{\mu\nu} = 0$ because $\nabla^\mu T_{\mu\nu} = 0$ (conservation of energy-momentum). This is analogous to $\partial_\mu F^{\mu\nu} = -J^\nu$ in EM leading to $\partial_\nu J^\nu = 0$

Einstein's Masterpiece

After years of effort, in late 1915, Einstein proposed the correct field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Where:

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the **Einstein Tensor**.
- $R_{\mu\nu}$ is the **Ricci Curvature Tensor**.
- $R = g^{\mu\nu}R_{\mu\nu}$ is the **Ricci Scalar**.
- $g_{\mu\nu}$ is the **Metric Tensor**.
- $T_{\mu\nu}$ is the **Stress-Energy Tensor**.
- G is Newton's gravitational constant, c is the speed of light.

Consistency: The covariant divergence of the Einstein Tensor is identically zero ($\nabla^\mu G_{\mu\nu} = 0$) due to the Bianchi identities, ensuring consistency with energy-momentum conservation ($\nabla^\mu T_{\mu\nu} = 0$).

Alternative Form and Implications

Alternative form of the EFE: Contracting the EFE with $g^{\mu\nu}$:

$$g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = g^{\mu\nu}\frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R - \frac{1}{2}(4)R = \frac{8\pi G}{c^4}T \quad \Longrightarrow \quad -R = \frac{8\pi G}{c^4}T$$

Substituting R back into the original EFE gives:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

This form is often very useful.

Vacuum Einstein Equations: When there is no source of matter or energy ($T^{\mu\nu} = 0$), the equations simplify to:

$$R_{\mu\nu} = 0$$

Important Note: $R_{\mu\nu} = 0$ does **not** imply that the spacetime is flat ($R_{\mu\nu\alpha\beta} = 0$). For example, Schwarzschild solution has $R_{\mu\nu} = 0$ but its spacetime is highly curved.

Detecting Curvature: Parallel Transport

- How do we know if space is intrinsically curved, not just appearing so due to coordinates?
- **Flat Space (e.g., 2D plane):** If you parallel transport a vector along a closed loop, it returns to its starting point with its original orientation.
- **Curved Space (e.g., 2D sphere):** If you parallel transport a vector along a closed loop on a sphere, it will generally return to its starting point with a different orientation. The change in orientation quantifies the curvature enclosed by the loop.
- This procedure defines the **Riemann Curvature Tensor**, which measures the amount of this "change upon parallel transport".

Change in a vector δV^ρ after parallel transport around an infinitesimal parallelogram defined by vectors A^μ and B^ν:

$$\delta V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

Special Relativity: Lorentz Transformations

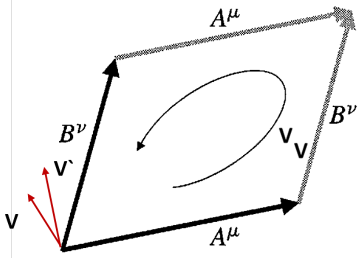


Figure: The change ocuurs in the vector V as a result of taking it around this loop is proportional to curvature tensor

Definition of the Riemann Curvature Tensor

The Riemann Curvature Tensor (also known as the Riemann-Christoffel tensor) is defined in terms of Christoffel symbols (Γ) and their derivatives:

$$R^{\sigma}_{\mu\nu\lambda} = \partial_{\nu}\Gamma^{\sigma}_{\mu\lambda} - \partial_{\lambda}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}$$

In 4-dimensions, this tensor has $4^4 = 256$ components, but due to its symmetries, only 20 are independent.

Properties of the Riemann Tensor:

1. **Anti-symmetry:**
    - In first two indices: $R_{\sigma\mu\nu\lambda} = -R_{\mu\sigma\nu\lambda}$
    - In last two indices: $R_{\sigma\mu\nu\lambda} = -R_{\sigma\mu\lambda\nu}$
2. **Symmetry:** $R_{\sigma\mu\nu\lambda} = R_{\nu\lambda\sigma\mu}$ (interchange of first pair and second pair)
3. **Cyclic Identity (First Bianchi Identity):**

$$R_{\sigma\mu\nu\lambda} + R_{\sigma\lambda\mu\nu} + R_{\sigma\nu\lambda\mu} = 0$$

Key Identities and Contractions

Bianchi Identity (Second Bianchi Identity): The covariant derivative of the Riemann tensor satisfies:

$$\nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\beta R_{\mu\nu\lambda\alpha} + \nabla_\alpha R_{\mu\nu\beta\lambda} = 0$$

This identity is crucial for the mathematical consistency of General Relativity.

Contractions of the Riemann Tensor:

- **Ricci Tensor ($R_{\mu\nu}$):** A symmetric rank (0,2) tensor, obtained by contracting the first and third indices of the Riemann tensor.

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Curvature Tensor: Bianchi Identity and Contractions

- **Ricci Scalar (R):** A scalar, obtained by contracting the Ricci tensor with the metric tensor.

$$R = g^{\mu\nu} R_{\mu\nu}$$

- **Contraction of Bianchi Identity:** This important contraction leads directly to the conservation law for the Einstein tensor:

$$\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$$

This ensures that the left side of the Einstein Field Equations (the Einstein Tensor) is covariantly conserved, matching the covariant conservation of the stress-energy tensor on the right side.