Schwarzschild Spacetime and Relativistic Orbits

Adel Awad

July 13, 2025

A Fundamental Solution in General Relativity

- In this chapter, we study a spherically symmetric, static, vacuum solution of Einstein Field Equations (EFE).
- Let's understand the meaning of these terms:
  1. **Vacuum Solution ($T_{\mu\nu} = 0$):** A solution without any source of matter or energy.
  2. **Spherically Symmetric:** The metric is invariant under rotations.
     - Quantities invariant under rotation: $\overline{x} \cdot \overline{x} = r^2$, $d\overline{x} \cdot d\overline{x}$, etc.
  3. **Static Solution:**
     - Metric components $g_{\mu\nu}$ do not depend on time.
     - No cross terms like $g_{0i} dt dx^i$ in the metric. This ensures invariance under time inversion symmetry ($t \to -t$).

Constructing the Metric

- Due to spherical symmetry, the metric depends only on dot products like $\overline{x} \cdot \overline{x} = r^2$, $\overline{x} \cdot d\overline{x}$, and $d\overline{x} \cdot d\overline{x}$.
- The most general form of the metric in Cartesian-like coordinates, respecting static and spherical symmetry, is:

$$dS^2 = -\tilde{A}(r)dt^2 + \tilde{C}(r)(\overline{x} \cdot d\overline{x})^2 + \tilde{D}(r)(d\overline{x} \cdot d\overline{x})$$

- Remember, the metric is always quadratic in the differentials (dt, dx^i).
- Converting to spherical coordinates ($x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$):
  - $\overline{x} \cdot d\overline{x} = rdr$
  - $d\overline{x} \cdot d\overline{x} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$
- Substituting these into the metric:

$$dS^2 = -\tilde{A}(r)dt^2 + \tilde{F}(r)dr^2 + \tilde{D}(r)(d\theta^2 + \sin^2\theta \; d\phi^2)$$

where $\tilde{F}(r) = r^2 \tilde{C}(r) + \tilde{D}(r)$.

Coordinate Transformation for Simplicity

- We can perform a coordinate transformation to simplify the $\tilde{D}(r)$ term. Let $r' = \sqrt{\tilde{D}(r)}$.
- After this transformation, and dropping the prime from r', the most general form for a spherically symmetric and static metric is:

$$dS^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \ d\phi^2)$$

- Our goal is to solve the Einstein Field Equations ($R_{\mu\nu} = 0$) for this metric to find the functions $A(r)$ and $B(r)$.

Components of the Metric Tensor

For the metric $dS^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2)$, the non-zero metric components are:

- $g_{00} = -A(r)$
- $g_{11} = B(r)$
- $g_{22} = r^2$
- $g_{33} = r^2 \sin^2\theta$

And their inverses ($g^{\mu\nu}$):

- $g^{00} = -A(r)^{-1}$
- $g^{11} = B(r)^{-1}$
- $g^{22} = r^{-2}$
- $g^{33} = (r^2 \sin^2\theta)^{-1}$

Connection Coefficients ($\Gamma^{\sigma}_{\mu\nu}$)

Using the formula $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$, the non-zero Christoffel symbols are:

- $\Gamma^{0}_{01} = \frac{A'}{2A}$
- $\Gamma^{1}_{00} = \frac{A'}{2B}$
- $\Gamma^{1}_{11} = \frac{B'}{2B}$
- $\Gamma^{1}_{22} = -\frac{r}{B}$
- $\Gamma^{1}_{33} = -\frac{r \sin^2\theta}{B}$
- $\Gamma^{2}_{12} = \frac{1}{r}$
- $\Gamma^{3}_{13} = \frac{1}{r}$
- $\Gamma^{3}_{23} = \cot\theta$
- $\Gamma^{2}_{33} = -\sin\theta\cos\theta$

(where $A' = \frac{dA}{dr}$ and $B' = \frac{dB}{dr}$).

Calculating $R_{\mu\nu}$ for Vacuum Solution

For a vacuum solution, we set $R_{\mu\nu} = 0$.

- R_{00} **component:**

$$R_{00} = \frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{(A')^2}{4AB} + \frac{A'}{rB}$$

- R_{11} **component:**

$$R_{11} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB}$$

- R_{22} **component:**

$$R_{22} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)$$

- R_{33} **component:**

$$R_{33} = \sin^2\theta \; R_{22}.$$

Finding $A(r)$ and $B(r)$

From the equations $R_{00} = 0$ and $R_{11} = 0$, after algebraic manipulation, one can derive:

$$\frac{A'}{A} + \frac{B'}{B} = 0$$

Integrating this gives $\ln A + \ln B = \text{const}$, or $AB = \text{const}$. Let $AB = C_0$.
So $B(r) = \frac{C_0}{A(r)}$.
Substituting this into the $R_{22} = 0$ equation:

$$1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0$$

$$1 - \frac{A}{C_0} - \frac{r}{2}\frac{A}{C_0}\left(\frac{A'}{A} - \frac{-C_0A'/A^2}{C_0/A}\right) = 0$$

$$1 - \frac{A}{C_0} - \frac{r}{2}\frac{A}{C_0}\left(\frac{A'}{A} + \frac{A'}{A}\right) = 0$$

$$1 - \frac{A}{C_0} - \frac{rA'}{C_0} = 0$$

Determining the Constants from Newtonian Limit

$$C_0 = A + rA' = \frac{d}{dr}(rA)$$

Integrating this last equation:

$$C_0 r + C_1 = rA(r) \implies A(r) = C_0 + \frac{C_1}{r}$$

And $B(r) = \frac{1}{1+C_1/(C_0 r)}$. Set $C_0 = 1$ (by scaling t).

In the Newtonian limit, we have $g_{00} \approx -1 - \frac{2\Phi}{c^2}$, where $\Phi = -\frac{GM}{r}$ is the Newtonian gravitational potential.

So, $C_1 = -2GM$.

The final form of the **Schwarzschild Metric** is:

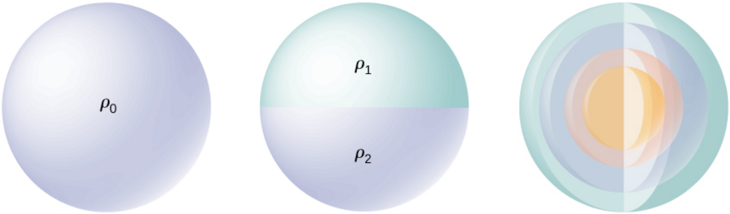
$$dS^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

This is the unique static, spherically symmetric, vacuum solution to Einstein's field equations.

Key Characteristics of the Solution

- **Asymptotically Flat:** As $r \to \infty$, the term $\left(1 - \frac{2GM}{r}\right) \to 1$.
  - Therefore, the metric approaches the Minkowski metric (flat spacetime). This means the gravitational field vanishes far from the source.
- **Dependence on Mass (M):**
  - The solution depends on only one parameter, M. If $M = 0$, the metric is again Minkowski (flat).
  - This M represents the total mass of the spherically symmetric object.
- **External Field Description:** This solution describes the gravitational field *outside* any spherically symmetric object, such as a star, planet, or black hole.
- **Independence of Mass Distribution (Birkhoff's Theorem):** The solution depends only on the total mass M, not on how the mass is distributed within the object (as long as it's spherically symmetric).
  - This is a powerful result: any spherically symmetric, vacuum solution must be the Schwarzschild solution, even if it's time-dependent.
  - Implication: A radially pulsating star does not emit gravitational waves in its vacuum exterior, as its metric must remain static (Schwarzschild).

Spherical Symmetry and Birkhoff's Theorem



(a) Spherically symmetric (b) Not spherically symmetric (c) Spherically symmetric

Schwarzschild Radius and Singularities

- One might notice two values of r where the metric is not mathematically well-defined:
  - $r_s = 2GM$ (Schwarzschild Radius)
  - $r_{sing} = 0$
- $r_s = 2GM$ **(Coordinate Singularity / Event Horizon):**
  - This was initially thought to be a non-physical region.
  - As r crosses r_s, the time and radial coordinates exchange their signature (e.g., $1 - \frac{2GM}{r} < 0$).
  - r_s is a **trapping surface** or **event horizon**: once you enter this region, you cannot go out, even at the speed of light.
  - It's a **coordinate singularity**, meaning no curvature component becomes infinite here. An observer crossing it might not notice a large change if M is large.
- $r = 0$ **(Curvature Singularity):**
  - At $r = 0$, Riemann Curvature components become infinite.
  - This is a **true physical singularity**.

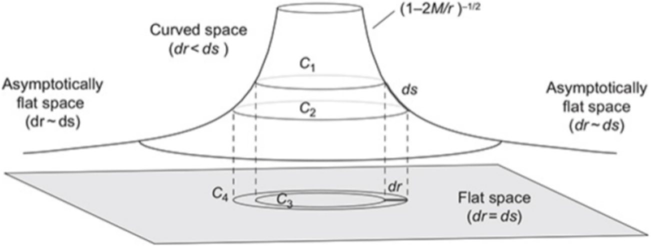
Proper Distance and Proper Time

- **Proper Radial Spatial Distance (S_{12}):**
  - Take $dt = 0, d\theta = 0, d\phi = 0$.
  - $dS^2 = \frac{dr^2}{1-\frac{2GM}{r}}$.
  - The proper distance between two points r_1 and r_2 is:

$$S_{12} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2GM}{r}}}$$

  - Notice how the combination $\frac{GM}{r}$ controls the proper distance. This distance is larger than the coordinate distance $r_2 - r_1$.
- **Proper Time ($d\tau$):**
  - For an observer at fixed spatial coordinates ($dr = 0, d\theta = 0, d\phi = 0$).
  - $dS^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r}\right) dt^2$.
  - So, $d\tau = \sqrt{1 - \frac{2GM}{r}} dt$.
  - **Time Dilation:** Proper time gets shorter as r approaches $2GM$. At $r = 2GM$, time appears to stop for an outside observer ($d\tau \to 0$ for finite dt).

Spherical Symmetry and Proper Distance



Frequency Shift in a Gravitational Field

- Consider an emitter at fixed spatial coordinates (r_E, θ_E, ϕ_E) emitting a photon, received by an observer at (r_R, θ_R, ϕ_R).
- The emission happens between t_E and $t_E + \Delta t_E$, and reception between t_R and $t_R + \Delta t_R$.
- Assuming emitter and receiver are static (following time-like geodesics, $dr = d\theta = d\phi = 0$):
  - Proper time interval at emitter: $\Delta \tau_E = \sqrt{1 - \frac{2GM}{r_E}} \Delta t_E$.
  - Proper time interval at receiver: $\Delta \tau_R = \sqrt{1 - \frac{2GM}{r_R}} \Delta t_R$.
- Since the time taken for light to travel between two fixed points is the same in coordinate time ($\Delta t_R = \Delta t_E$):

$$\frac{\Delta \tau_R}{\Delta \tau_E} = \frac{\sqrt{1 - \frac{2GM}{r_R}}}{\sqrt{1 - \frac{2GM}{r_E}}}$$

Frequency Shift in a Gravitational Field

- Since frequency $\nu \propto \frac{1}{\tau}$:

$$\frac{\nu_R}{\nu_E} = \frac{\sqrt{1 - \frac{2GM}{r_E}}}{\sqrt{1 - \frac{2GM}{r_R}}}$$

- For a receiver at $r_R = \infty$ (far away) and an emitter at $r_E = r$:

$$\nu_R = \sqrt{1 - \frac{2GM}{r}}\nu_E$$

- **Redshift:** If $r_E < r_R$, then $\nu_R < \nu_E$, meaning the frequency is shifted to red (lower frequency).
- **Extreme Redshift:** As the signal is emitted from $r = r_s = 2GM$, the frequency $\nu_R \to 0$, meaning the signal effectively disappears.

Classical Orbits in a Central Force

- For a particle moving in a central force (e.g., gravitational field), the energy E is:

$$E = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} + U(r)$$

where l is the angular momentum, and $U(r)$ is the potential energy.

- For Newtonian gravity, $U(r) = -\frac{GM}{r}$ (assuming unit mass for the particle).
- The **Newtonian Effective Potential** is:

$$V_{eff}^{(N)} = -\frac{GM}{r} + \frac{l^2}{2r^2}$$

- The radial equation of motion is $\dot{r} = \pm\sqrt{2(E - V_{eff}^{(N)})}$.
- The shape of the orbit $\phi(r)$ is found by integrating:

$$\phi(r) = \int \frac{(l/r^2)}{\sqrt{2(E - V_{eff}^{(N)})}} dr$$

Energy and Angular Momentum

- Newtonian orbits are conic sections: Hyperbola ($E > 0$), Parabola ($E = 0$), Ellipse ($V_{min} < E < 0$), Circle ($E = V_{min}$).
- For a test particle in Schwarzschild spacetime, the Lagrangian is $L = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$.
- From Euler-Lagrange equations, symmetries (Killing vectors) lead to conserved quantities:
  - **Energy (E):** Due to time-translation symmetry ($\frac{\partial L}{\partial t} = 0$).

$$E = -\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}$$

    (Here, $d\tau$ is proper time). This is $E = -g_{00}u^0$.
  - **Angular Momentum (l):** Due to azimuthal symmetry ($\frac{\partial L}{\partial \phi} = 0$).

$$l = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

    For equatorial orbits ($\theta = \pi/2$), this simplifies to $l = r^2 \frac{d\phi}{d\tau}$.

Energy and Angular Momentum

- The normalization condition for the 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ is $u_\mu u^\mu = -1$:

$$-1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 +$$

$$+ r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2$$

Equation of Motion and GR Effective Potential

- For equatorial orbits ($\theta = \pi/2$, so $d\theta/d\tau = 0$), substituting the conserved quantities (E and l) into the 4-velocity normalization condition, we can rearrange to get an energy-like equation:

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - \frac{2GM}{r}\right)$$

- This can be written in the form $\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}^{(GR)}(r) = \frac{E^2-1}{2} = \mathcal{E}$, where $\mathcal{E}$ is constant.
- The **General Relativistic Effective Potential** is:

$$V_{eff}^{(GR)}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$$

- **Comparison with Newtonian:** The GR effective potential has an additional attractive term $-\frac{GMl^2}{r^3}$, which is proportional to $1/r^3$. This term becomes significant at small radii and leads to important new phenomena.

Equation of Motion and GR Effective Potential

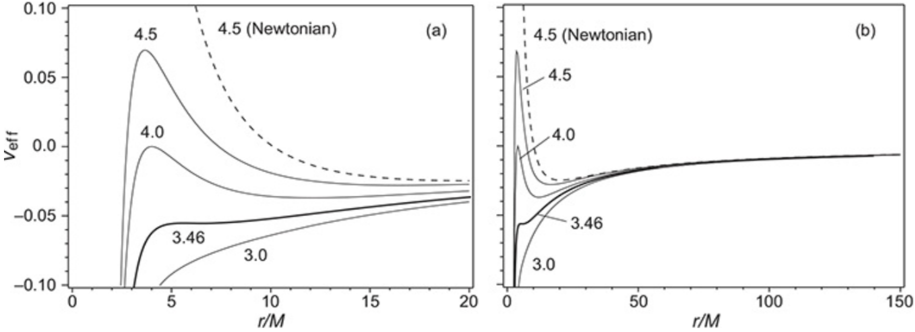
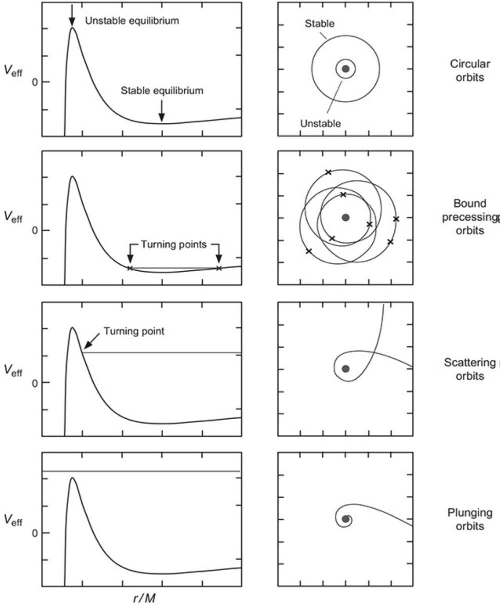


Figure: Different values for l/m=4.5,4.0,3.46,3.0

Impact of the $1/r^3$ Term

- The $1/r^3$ term in $V_{eff}^{(GR)}$ causes the potential to drop more steeply at small r compared to the Newtonian case.
- This leads to:
  - A maximum in the potential at small radii, corresponding to unstable circular orbits.
  - A minimum at larger radii, corresponding to stable circular orbits.
  - For very small r, the potential plunges to $-\infty$, indicating that particles eventually fall into the singularity.
- Figure illustrates the different types of orbits based on the energy $\mathcal{E}$ relative to the effective potential.
  - **Circular Orbits:** Occur at extrema of V_{eff}.
  - **Bound Precessing Orbits:** Energy between minimum and maximum, leads to non-closed elliptical-like orbits that precess.
  - **Scattering Orbits:** Energy above the maximum, particles approach and then recede, but are deflected.
  - **Plunging Orbits:** Energy below the minimum, particles fall directly into the central mass.

GR Effective Potential: Different Orbits



Conditions and Properties

- Circular orbits occur when the energy $\mathcal{E}$ is equal to a minimum or maximum of the effective potential ($V_{eff}^{(GR)}$).
- At these points, $\dot{r} = 0$, so $\mathcal{E} = V_{eff}^{(GR)}$. Also, $\frac{dV_{eff}^{(GR)}}{dr} = 0$.
- Solving $\frac{dV_{eff}^{(GR)}}{dr} = 0$ for r (for circular orbits) yields:

$$r = \frac{l^2}{2GM} \pm \frac{1}{2}\sqrt{\frac{l^4}{(GM)^2} - 12l^2}$$

- This equation has two solutions if $l^2/(GM)^2 > 12$:
  - The plus sign corresponds to a stable circular orbit (minimum of potential).
  - The minus sign corresponds to an unstable circular orbit (maximum of potential).
- The innermost stable circular orbit (ISCO) is at $r = 6GM$. Orbits inside this radius are unstable.

Kepler's Laws in General Relativity

- The angular velocity (Ω) of a particle in a circular orbit, as seen by a distant observer (for $\theta = \pi/2$), is:

$$\Omega = \frac{d\phi}{dt} = \frac{l}{r^2}\left(1 - \frac{2GM}{r}\right)\frac{1}{E}$$

- Using the relationship between E, l, and r for circular orbits, this simplifies to:

$$\Omega = \sqrt{\frac{GM}{r^3}}$$

- This is precisely **Kepler's Third Law** from Newtonian gravity! It shows that for circular orbits, the orbital period is the same as in Newtonian gravity, even though the underlying mechanics are different due to spacetime curvature.
- The period P is:

$$P = \frac{2\pi}{\Omega}$$