Introduction to General Relativity

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Contents

- **Topics:**
  1. **Special Relativity and Spacetime**
  2. **Equivalence Principle and Bending of light**
  3. **Tensors and Covariant Derivative**
  4. **Geodesic Equations and Field Equations**
  5. **GR and Cosmology**
  6. **Spherically Symmetric Solutions and Stars**

Special Relativity: The Foundation

- **Core Principles:**
  1. **Constant Speed of Light:** The speed of light in a vacuum (c) is constant in all inertial reference frames (IRF).
  2. **Relativity Principle:** The laws of physics have the same form in all inertial reference frames.
- **Implications:** Space and time are not absolute but are intertwined into a single entity called spacetime. Time is relative, space is relative and mass is energy.

Special Relativity: Lorentz Transformations

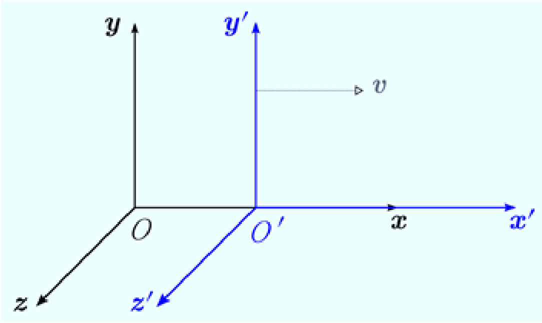


Figure: Two inertial frames: Σ' is moving relative to Σ

Lorentz Transformations

Consider two IRFs, Σ and Σ', where Σ' moves with uniform velocity $\vec{v} = v\hat{x}$ relative to Σ. The relations between coordinates are

$$\begin{aligned} ct' &= \gamma(ct - \beta\, x) \\ x' &= \gamma(x - \beta\, ct) \\ y' &= y \\ z' &= z \end{aligned}$$

Where $\beta = \frac{v}{c}$ and $\gamma = (1 - \beta^2)^{-1/2}$ (Lorentz factor).

Matrix Form ($x'^\mu = \Lambda^\mu{}_\nu x^\nu$):

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Here, $x^\mu \equiv (ct, x, y, z)$ is a 4-vector (using ct for time component for consistent units).

Lorentz transformations leave the following quantity invariant:

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2 = S^2$$

The nature of this interval classifies event relationships:

- $S^2 < 0$: **Time-like interval**.
  - Temporal separation $>$ spatial separation.
  - Events are causally connected ($v < c$). An object can travel between them.
- $S^2 = 0$: **Null-like interval**.
  - Events are connected by a light ray ($v = c$).
- $S^2 > 0$: **Space-like interval**.
  - Spatial separation $>$ temporal separation.
  - Events are not causally connected ($v > c$ would be required).

Special Relativity: Light Cone

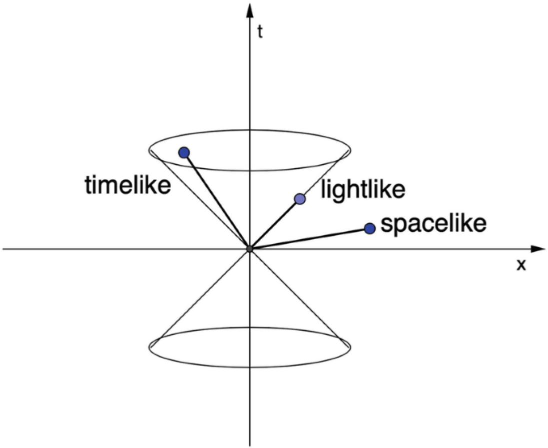


Figure: timelike, spacelike and lightlike intervals

Spacetime Four-Vector

- **Concept:** In Special Relativity, space and time are unified into a single four-dimensional entity called **spacetime**. Events are points in this spacetime.
- **Position Four-Vector (x^μ):** Combines the time coordinate (multiplied by c) and the three spatial coordinates into a single vector.
  - $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$
  - Where μ is an index ranging from 0 to 3.
- **Lorentz Invariance:** The "length" or "interval" between two events in spacetime is invariant (the same for all inertial observers), even though individual space and time components change.
  - $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ (Minkowski metric)
- **Differential Operator:** $\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

  Also, $\partial_\mu \partial^\mu = -{\partial_0}^2 + {\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2 = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Four-Momentum

- **Concept:** Just as space and time are unified, energy and momentum are also combined into a single four-vector in Special Relativity.
- **Momentum Four-Vector (p^μ):** Combines the relativistic energy (divided by c) and the three spatial momentum components.
    - $p^\mu = (p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$
    - Where $E = \gamma mc^2$ is the relativistic energy and $\mathbf{p} = \gamma m\mathbf{v}$ is the relativistic momentum.
- **Lorentz Invariance:** The "length squared" of the four-momentum vector is also invariant and related to the rest mass of the particle:
    - $-(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -(E/c)^2 + |\mathbf{p}|^2 = -(mc)^2$
    - This leads directly to the famous mass-energy equivalence $E^2 = (pc)^2 + (mc^2)^2$.
- **Significance:** Writing down laws of physics using 4-vectors or scalar ensures that the physics is covariant under Lorentz transformations.

Maxwell's Equations and Relativity

- **The Challenge:** Maxwell's equations were inherently consistent with the second postulate of special relativity (constant speed of light). However, their form was not obviously invariant under Galilean transformations, implying a preferred reference frame (the aether).
- **Einstein's Insight:** Special Relativity showed that Maxwell's equations are *already* relativistically correct. The problem was with Newtonian mechanics, not with electromagnetism.
- **The Unification:** Special Relativity provided the framework to show that electric and magnetic fields are not independent entities but are two aspects of a single **electromagnetic field**, whose appearance depends on the observer's frame of reference.

Introducing the Electromagnetic Field Tensor

- Need for Covariant Form: To express Maxwell's equations in a way that is explicitly invariant under Lorentz transformations, we use **tensor notation**. This makes the relativistic nature of the equations transparent.
- We introduce a four-potential $A_\mu = (\phi/c, \mathbf{A})$ and a 2nd rank anti-symmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
- Electromagnetic Field Tensor ($F^{\mu\nu}$): It combines the electric field ($\mathbf{E}$) and magnetic field ($\mathbf{B}$) into a single mathematical object.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Maxwell's Equations in Covariant Form

- **Two Tensor Equations:** The four original Maxwell's equations can be elegantly expressed as two tensor equations:
  1. **First Pair (Homogeneous Equations):** Describes the absence of magnetic monopoles and Faraday's law of induction.
  $$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$
  2. **Second Pair (Inhomogeneous Equations):** Describes Gauss's law and Ampere-Maxwell's law.
  $$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$
     - J^ν: Four-current density vector $(c\rho, \mathbf{J})$.
     - μ_0: Permeability of free space.
- **Significance:** These covariant forms explicitly demonstrate the Lorentz invariance of Maxwell's equations, confirming their consistency with Special Relativity.

Inertial vs. Gravitational Mass

- **Inertial Mass (m_I):** Defined by Newton's $F = m_I a$. Resistance to acceleration.
- **Gravitational Mass (m_G):** Defined by Newton's $F_G = m_G \frac{M_G G_N}{r^2}$. Source of, or response to, gravitational field.
- **Empirical Fact:** $m_I = m_G$. This leads to gravitational acceleration $a_G = \frac{G_N M_G}{r^2}$ being independent of the object's mass or composition.
- **Key Insight:** Gravity is "blind" to the physical content of any object. This suggests gravity might be a manifestation of spacetime geometry rather than a force since all objects are affected in the same way.

Gravity as a spacetime curvature

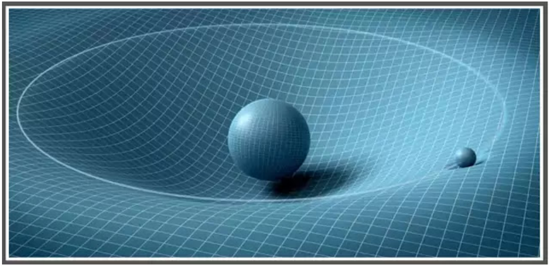
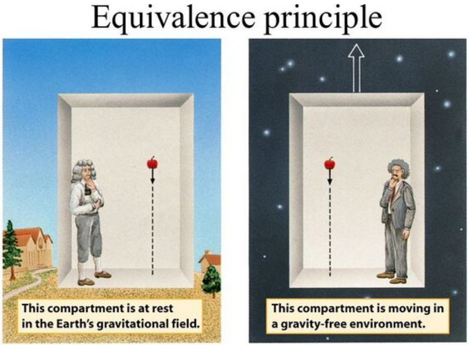


Figure: Gravitational acceleration does not depend on the mass of the falling object!

Einstein's Elevator Experiment

- An observer in a windowless elevator cannot distinguish between:
  - Being in a uniformly accelerating rocket in deep space.
  - Being at rest in a uniform gravitational field.
- Similarly, free-falling in a gravitational field is locally indistinguishable from being in deep space (weightlessness).
- **Conclusion (Equivalence Principle):** Uniform acceleration is locally equivalent to a uniform gravitational field.
- **Consequence:** Gravity affects the very fabric of spacetime, causing it to bend or curve. This curvature dictates the paths objects take, replacing the classical "force" of gravity.

Equivalence Principle



(a) The apple hits the floor of the compartment because the Earth's gravity accelerates the apple downward.

(b) The apple hits the floor of the compartment because the compartment accelerates upward.

Special Relativity: Lorentz Transformations

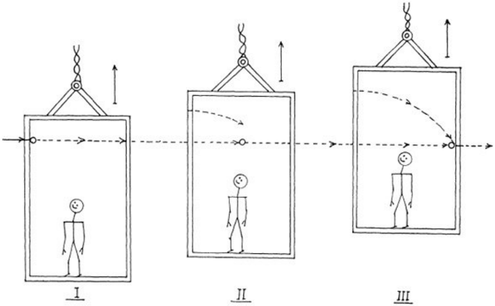


Figure: If uniform acceleration affects light gravity must affect it in the same way!

Why Tensors?

- General Relativity describes gravity in **general (arbitrary) reference frames**, not just inertial ones (like SR).
- We need mathematical quantities whose transformation laws are well-defined under general coordinate transformations: $x^\mu \equiv (x^0, x^1, x^2, x^3) \longrightarrow x'^\mu \equiv (x'^0, x'^1, x'^2, x'^3)$.
- Unlike SR's linear Lorentz transformations ($A'^\mu = \Lambda^\mu{}_\nu A^\nu$), GR uses non-linear transformations where derivatives like $\frac{\partial x'^\mu}{\partial x^\nu}$ become crucial.

Definition: A tensor is a geometric object that transforms in a well-defined way under coordinate transformations.

Types of Tensors

1. Rank (0,0) Tensor (Scalar): A quantity invariant under coordinate transformations.

$$\phi'(x') = \phi(x)$$

   Examples: Spacetime interval, mass, temperature.

2. Rank (1,0) Tensor (Contravariant Vector): Transforms like coordinates (dx^μ).

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

   Examples: 4-position, 4-velocity.

3. Rank (0,1) Tensor (Covariant Vector): Transforms like gradients ($\frac{\partial}{\partial x^\mu}$).

$$A'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} A_\mu$$

   Examples: Gradient of a scalar field, 4-momentum with lower index.

Types of Tensors

General Rank (k, l) **Tensor:** k upper (contravariant) indices, l lower (covariant) indices.

$$T'^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial x'^{\nu_l}} \right) T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l}$$

Properties of Tensors

- **Contraction:** Summing over one contravariant and one covariant index of a tensor reduces its rank by 2. Example: Contraction of A'^{μ} and B'_{μ} forms a scalar:

$$A'_{\mu}B'^{\mu} = \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}A_{\alpha}\right)\left(\frac{\partial x'^{\mu}}{\partial x^{\beta}}B^{\beta}\right) = \frac{\partial x^{\alpha}}{\partial x^{\beta}}A_{\alpha}B^{\beta} = \delta^{\alpha}_{\beta}A_{\alpha}B^{\beta} = A_{\alpha}B^{\alpha}$$

- **Symmetric and Anti-symmetric Tensors:**
  - Symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$
  - Anti-symmetric: $T^{\alpha\beta} = -T^{\beta\alpha}$
- **Decomposition:** Any tensor $B_{\mu\nu}$ can be uniquely decomposed into symmetric and anti-symmetric parts:

$$B_{\mu\nu} = B_{(\mu\nu)} + B_{[\mu\nu]}$$

where $B_{(\mu\nu)} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$ and $B_{[\mu\nu]} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$.

The Fundamental Role of $g_{\mu\nu}$

- The **metric tensor** ($g_{\mu\nu}$) is central to General Relativity.
- It defines infinitesimal distances between any two points in curved spacetime. It is a generalization of the Pythagorean theorem.

The square of an infinitesimal spacetime interval dS^2:

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + 2g_{01} dx^0 dx^1 + g_{22}(dx^2)^2 + \dots$$

In flat spacetime, $g_{\mu\nu}$ reduces to the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ (using $(-, +, +, +)$ signature).

Raising and Lowering Indices: The metric tensor and its inverse ($g^{\mu\nu}$) are used to transform between covariant and contravariant forms of tensors:

- Lowering: $A_\mu = g_{\mu\nu} A^\nu$
- Raising: $A^\mu = g^{\mu\nu} A_\nu$
- Example: $g_{\nu\alpha} T^{\mu\alpha} = T^\mu_\nu$

The Problem with Partial Derivatives

- Partial derivatives of scalars (ϕ) are tensors: $\frac{\partial \phi'}{\partial x'^{\mu}} = \frac{\partial \phi}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}$.
- However, the partial derivative of a vector $A^{\mu}(x)$ **does not transform as a tensor**:

$$\frac{\partial A'^{\mu}}{\partial x'^{\nu}} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial A^{\alpha}}{\partial x^{\beta}} + \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x^{\alpha}} A^{\alpha}$$

The second term, involving second partial derivatives of the coordinate transformation, prevents it from being a tensor. This highlights that basis vectors change in curved spacetime.

Covariant Derivative

Consider a vector $V = V^\mu \hat{e}_\mu$. Its total variation δV includes changes in components and changes in basis vectors:

$$\delta V = \delta V^\mu \hat{e}_\mu + V^\mu \delta \hat{e}_\mu$$

The change in basis vectors can be expressed using Christoffel symbols:

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \Gamma^\alpha_{\mu\nu} \hat{e}_\alpha$$

Substituting this into the derivative of V with respect to x^β:

$$D_\beta V = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\nu} V^\mu \frac{\partial x^\nu}{\partial x^\beta} \right) \hat{e}_\alpha = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right) \hat{e}_\alpha$$

The term in parentheses is precisely the definition of the covariant derivative $\nabla_\beta V^\alpha$. It accounts for the non-Euclidean nature of parallel transporting a vector in curved space.

Covariant Derivative

To define differentiation that yields a tensor in curved spacetime, we introduce the **covariant derivative**, ∇_ν.

- For a contravariant vector A^μ:

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\alpha\nu} A^\alpha$$

- For a covariant vector A_μ:

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\alpha$$

Christoffel Symbol ($\Gamma^\sigma_{\mu\nu}$): Not a tensor, but represents how basis vectors change in curved space. It is calculated from the metric tensor:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

A crucial property: $\nabla_\alpha g_{\mu\nu} = 0$ (metric is covariantly constant).

Generalizing Straight Lines to Curved Spacetime

- In Euclidean space, a straight line is the shortest path between two points, and its tangent vector does not change direction.
- A **geodesic** is the generalization of a straight line to curved spaces or spacetimes.
- It is the curve of extremal (minimum or maximum) length (interval) between two points.
- Equivalently, it's the path along which a particle moves when only acted upon by gravity (i.e., its 4-velocity is parallel transported along its path).

3D Curve Example: A curve $\overline{X}(s) = X(s)\hat{e}_x + Y(s)\hat{e}_y + Z(s)\hat{e}_z$ has a tangent vector $\overline{t} = \frac{d\overline{x}}{ds}$.

The Geodesic Equation

From Parallel Transport: For a geodesic, the covariant derivative of the tangent vector $t^\alpha = \frac{dx^\alpha}{ds}$ along the curve must be zero (or proportional to itself for a non-affine parameter).

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \lambda(s) \left(\frac{dx^\alpha}{ds} \right)$$

If we can find a parameter s' such that $\lambda(s') = 0$, it's called an **affine parameter** (e.g., proper time for massive particles). The equation becomes:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

This is the geodesic equation.

From Variational Principle (Euler-Lagrange): Consider a free particle with Lagrangian $L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$. Applying the Euler-Lagrange equations $\frac{d}{ds} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0$ yields the same geodesic equation:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

Relativistic Kinematics and Charge Conservation

- **4-Velocity (u^μ):** For a massive particle, in its rest frame, time is measured by proper time τ. The spacetime interval is $dS^2 = -d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

$$-1 = \eta_{\mu\nu} \left(\frac{dx^\mu}{d\tau}\right) \left(\frac{dx^\nu}{d\tau}\right)$$

  The 4-velocity is $u^\mu = \frac{dx^\mu}{d\tau}$. Its components are $u^\mu \equiv (\gamma, \gamma v^i)$.
- **4-Current (J^μ):** Generalizes charge density and current to spacetime.
  - Classical continuity equation: $\frac{\partial \rho}{\partial t} + \overline{\nabla} \cdot \overline{J} = 0$.
  - Relativistic charge density: $\rho = \rho_0 \gamma$ (where ρ_0 is rest charge density).
  - 4-current: $J^\mu \equiv (\rho_0 \gamma, \rho_0 \gamma \overline{v}) = \rho_0 u^\mu$ (using $c = 1$ in u^0 for proper units).
  - Covariant continuity equation: $\partial_\mu J^\mu = 0$. This implies charge conservation.

Energy and Momentum Distribution

- For matter particles with rest mass m_0, the rest mass density is $\rho_0 = m_0 n$.
- The distribution of energy and momentum in spacetime is described by the **Stress-Energy Tensor** (or Energy-Momentum Tensor), $T^{\mu\nu}$.
- For a simple collection of dust particles (no pressure):

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu$$

- **Components and Physical Interpretation:**
  - $T^{00} = \rho_0 (u^0)^2 = \rho_0 \gamma^2$: Energy/mass density.
  - $T^{0i} = \rho_0 u^0 u^i = \rho_0 \gamma u^i$: Energy flux in i-direction (or mass current).
  - $T^{i0} = \rho_0 u^i u^0 = \rho_0 \gamma u^i$: Momentum density in i-direction (or energy flux).
  - $T^{ij} = \rho_0 u^i u^j$: Momentum flux (stress or pressure).

Perfect Fluid Stress-Energy Tensor

- For a perfect fluid (isotropic, no viscosity, no heat conduction) with energy density ρ and isotropic pressure P:

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + \eta^{\mu\nu}P \quad \text{(in SR)}$$

- In curved spacetime, the flat metric $\eta^{\mu\nu}$ is replaced by the spacetime metric $g^{\mu\nu}$:

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + g^{\mu\nu}P \quad \text{(in GR)}$$

- **Conservation of Energy and Momentum:** In GR, this is expressed by the covariant divergence of the stress-energy tensor being zero: $\nabla_{\mu}T^{\mu\nu} = 0$.

Connecting to Classical Gravity

- **Classical Gravity (Newtonian):**
  - Poisson's equation (field equation): $\nabla^2\phi = 4\pi G\rho$.
  - Newton's second law (test particle equation): $\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i}$.
- In GR, Newton's second law for a test particle should emerge from the geodesic equation in the following limits:
  - **Weak field:** $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($h_{\mu\nu}$ is small).
  - **Non-relativistic:** Particle velocities $v \ll c$, so $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \simeq c$.
  - **Static:** $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$.
- The geodesic equation: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$. In the weak-field, non-relativistic, static limit, the dominant term is for $\alpha = \beta = 0$:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0$$

From Geodesics to Potential

Since $\frac{dx^0}{d\tau} \approx c$, we have $\frac{d^2x^\mu}{d\tau^2} + c^2\Gamma^\mu_{00} = 0$.
Evaluating Γ^μ_{00}:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{0\alpha} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{\mu\alpha}\frac{\partial g_{00}}{\partial x^\alpha}$$

Using $g^{\mu\alpha} \approx \eta^{\mu\alpha}$ and $g_{00} = \eta_{00} + h_{00} = -1 + h_{00}$:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}\eta^{\mu\alpha}\frac{\partial h_{00}}{\partial x^\alpha}$$

For $\mu = i$ (spatial component):

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}$$

Comparing with Newton's second law ($\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x^i}$, with $d\tau \approx dt$):

$$h_{00} = -\frac{2\phi}{c^2} \implies g_{00} = -1 - \frac{2\phi}{c^2}$$

From Poisson's Equation to a Tensor Form

Substitute g_{00} into Poisson's equation $\nabla^2 \phi = 4\pi G \rho$:

$$-\frac{c^2}{2}\nabla^2 g_{00} = 4\pi G \rho = \frac{4\pi G}{c^2} T_{00}$$

$$\nabla^2 g_{00} = -\frac{8\pi G}{c^4} T_{00}$$

This suggests a generalization for a full covariant field equation:

$$\nabla^2 g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

However, this is **not correct** because $\nabla_\alpha g_{\mu\nu} = 0$ (covariant derivative of metric is zero). The left side would be zero, but $T_{\mu\nu}$ is generally non-zero.

Guiding Principle for EFE: We need an equation of the form $X_{\mu\nu} = K T_{\mu\nu}$ such that $\nabla^\mu X_{\mu\nu} = 0$ because $\nabla^\mu T_{\mu\nu} = 0$ (conservation of energy-momentum). This is analogous to $\partial_\mu F^{\mu\nu} = -J^\nu$ in EM leading to $\partial_\nu J^\nu = 0$.

Detecting Curvature: Parallel Transport

- How do we know if space is intrinsically curved, not just appearing so due to coordinates?
- **Flat Space (e.g., 2D plane):** If you parallel transport a vector along a closed loop, it returns to its starting point with its original orientation.
- **Curved Space (e.g., 2D sphere):** If you parallel transport a vector along a closed loop on a sphere, it will generally return to its starting point with a different orientation. The change in orientation quantifies the curvature enclosed by the loop.
- This procedure defines the **Riemann Curvature Tensor**, which measures the amount of this "change upon parallel transport".

Transporting a Vector in a Closed Loop

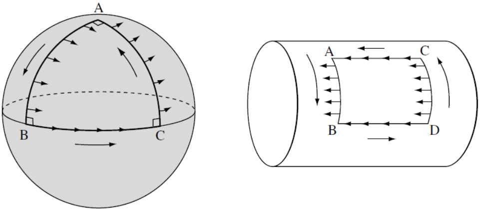


Figure: Transporting a vector around a closed loop as a measure of curvature tensor

Transporting a Vector in a Closed Loop

The change in a vector δV^ρ after parallel transport around an infinitesimal parallelogram defined by vectors A^μ and B^ν:

$$\delta V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

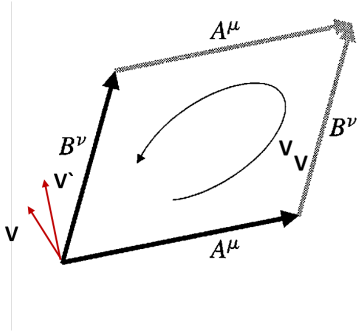


Figure: The change occurs in the vector V as a result of taking it around this loop is proportional to curvature tensor

Definition of the Riemann Curvature Tensor

The Riemann Curvature Tensor (also known as the Riemann-Christoffel tensor) is defined in terms of Christoffel symbols (Γ) and their derivatives:

$$R^{\sigma}_{\mu\nu\lambda} = \partial_{\nu}\Gamma^{\sigma}_{\mu\lambda} - \partial_{\lambda}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}$$

In 4-dimensions, this tensor has $4^4 = 256$ components, but due to its symmetries, only 20 are independent.

Properties of the Riemann Tensor:

1. **Anti-symmetry:**
   - In first two indices: $R_{\sigma\mu\nu\lambda} = -R_{\mu\sigma\nu\lambda}$
   - In last two indices: $R_{\sigma\mu\nu\lambda} = -R_{\sigma\mu\lambda\nu}$
2. **Symmetry:** $R_{\sigma\mu\nu\lambda} = R_{\nu\lambda\sigma\mu}$ (interchange of first pair and second pair)
3. **Cyclic Identity (First Bianchi Identity):**

$$R_{\sigma\mu\nu\lambda} + R_{\sigma\lambda\mu\nu} + R_{\sigma\nu\lambda\mu} = 0$$

Key Identities and Contractions

Bianchi Identity (Second Bianchi Identity): The covariant derivative of the Riemann tensor satisfies:

$$\nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\beta R_{\mu\nu\lambda\alpha} + \nabla_\alpha R_{\mu\nu\beta\lambda} = 0$$

This identity is crucial for the mathematical consistency of General Relativity.

Contractions of the Riemann Tensor:

- **Ricci Tensor ($R_{\mu\nu}$):** A symmetric rank (0,2) tensor, obtained by contracting the first and third indices of the Riemann tensor.

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Curvature Tensor: Bianchi Identity and Contractions

- **Ricci Scalar (R):** A scalar, obtained by contracting the Ricci tensor with the metric tensor.

$$R = g^{\mu\nu} R_{\mu\nu}$$

- **Contraction of Bianchi Identity:** This important contraction leads directly to the conservation law for the Einstein tensor:

$$\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$$

This ensures that the left side of the Einstein Field Equations (the Einstein Tensor) is covariantly conserved, matching the covariant conservation of the stress-energy tensor on the right side.

Einstein's Masterpiece

After years of effort, in late 1915, Einstein proposed the correct field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where:

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the **Einstein Tensor**.
- $R_{\mu\nu}$ is the **Ricci Curvature Tensor**.
- $R = g^{\mu\nu} R_{\mu\nu}$ is the **Ricci Scalar**.
- $g_{\mu\nu}$ is the **Metric Tensor**.
- $T_{\mu\nu}$ is the **Stress-Energy Tensor**.
- G is Newton's gravitational constant, c is the speed of light.

Consistency: The covariant divergence of the Einstein Tensor is identically zero ($\nabla^\mu G_{\mu\nu} = 0$) due to the Bianchi identities, ensuring consistency with energy-momentum conservation ($\nabla^\mu T_{\mu\nu} = 0$).

Alternative Form and Implications

Alternative form of the EFE: Contracting the EFE with $g^{\mu\nu}$:

$$g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = g^{\mu\nu}\frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R - \frac{1}{2}(4)R = \frac{8\pi G}{c^4}T \quad \Longrightarrow \quad -R = \frac{8\pi G}{c^4}T$$

Substituting R back into the original EFE gives:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

This form is often very useful.

Vacuum Einstein Equations: When there is no source of matter or energy ($T^{\mu\nu} = 0$), the equations simplify to:

$$R_{\mu\nu} = 0$$

Important Note: $R_{\mu\nu} = 0$ does **not** imply that the spacetime is flat ($R_{\mu\nu\alpha\beta} = 0$). For example, Schwarzschild solution has $R_{\mu\nu} = 0$ but its spacetime is highly curved.