Introduction to Cosmology

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The Universe as We Observe It

- One can observe that the Universe as we see it is:
  - **Isotropic:** Looks the same in every direction (no preferred direction).
  - **Homogeneous:** Looks the same in every point or region (no preferred point/region).
- These two assumptions are fundamental in modern cosmology.

The Metric of an Isotropic and Homogeneous Universe

- The assumptions of isotropy and homogeneity lead to a specific form of the spacetime metric:

$$dS^2 = -dt^2 + a^2(t)d\sigma^2$$

- There are no cross terms such as dtdx.
- $a(t)$ is the **scale factor**, which depends on time.
- $d\sigma^2$ is the spatial metric on a 3-dimensional hypersurface Σ.
- The entire spacetime can be viewed as $R \times \Sigma$ (Time (1D) $\times$ Space (3D)).

The Spatial Metric $d\sigma^2$

- We found that the spatial metric $d\sigma^2$ has the form $\gamma_{ij}(x_i)dx^i dx^j$ (where x_i are dimensionless coordinates).
- In spherical coordinates, $d\sigma^2$ is given by:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2(d\theta^2 + \sin^2\theta \; d\phi^2)$$

- Here, k is the **curvature parameter**:
  - $k = 0$: flat space.
  - $k = -1$: hyperbolic (open) 3-dimensional surface.
  - $k = +1$: spherical (closed) 3-dimensional surface.

Possible Spacial Metrics

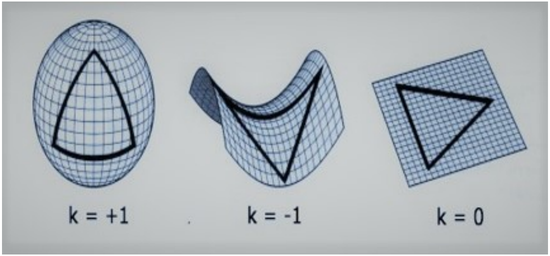

Figure: Spacial metrics with negative, positive and zero curvature!

Standardizing the Metric Form

- This metric is the standard form of the **Robertson-Walker metric**:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right]$$

- In order to solve Einstein Field Equations for this metric we must start with calculating connection components.

Christoffel Symbols for RW Metric

For the Robertson-Walker metric, the non-vanishing Christoffel symbols (Connection Components) are:

- $\Gamma^0_{11} = \frac{a\dot{a}}{1-Kr^2}$
- $\Gamma^0_{22} = a\dot{a}r^2$
- $\Gamma^0_{33} = a\dot{a}r^2 \sin^2\theta$
- $\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}$
- $\Gamma^1_{11} = \frac{Kr}{1-Kr^2}$
- $\Gamma^1_{22} = -r(1-Kr^2)$
- $\Gamma^1_{33} = -r(1-Kr^2)\sin^2\theta$
- $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$
- $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$
- $\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$
- $\Gamma^2_{33} = -\sin\theta\cos\theta$

Ricci Tensor for RW Metric

The non-vanishing components of the Ricci Tensor ($R_{\mu\nu}$) for the Robertson-Walker metric are:

- $R_{00} = -3\frac{\ddot{a}}{a}$
- $R_{11} = \frac{a\ddot{a}+2\dot{a}^2+2K}{1-Kr^2}$
- $R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)$
- $R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)\sin^2\theta$

Modeling Matter Content

- The energy-momentum conservation is given by $\nabla_\mu T^{\mu\nu} = 0$.
- In the case of isotropy and homogeneity, matter is represented by a **perfect fluid**.
- The stress-energy tensor for a perfect fluid is:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + Pg_{\mu\nu}$$

- Here, $U^\mu = (1, 0, 0, 0)$ is the 4-velocity of any material point of this perfect fluid (in the comoving frame).
- The components of the stress-energy tensor in the comoving frame are:

$$T^\mu_\nu = \text{diag}(-\rho, P, P, P)$$

- The trace of the stress-energy tensor is $T = T^\mu_\mu = 3P - \rho$.

Connecting Curvature to Matter Content

The Einstein Field Equations are given by:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

- **00-component of EFE:**

$$R_{00} = 8\pi G(T_{00} - \frac{1}{2}g_{00}T)$$

Substituting $R_{00} = -3\frac{\ddot{a}}{a}$ and $T_{00} = \rho$, $g_{00} = -1$:

$$-3\frac{\ddot{a}}{a} = 8\pi G(\rho - \frac{1}{2}[-\rho + 3P])$$

$$-3\frac{\ddot{a}}{a} = 8\pi G(\frac{3}{2}\rho + \frac{3}{2}P)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

This is the **second Friedmann equation** (also known as the acceleration equation).

The First Friedmann Equation

- **22-component of EFE:**

$$R_{22} = 8\pi G(T_{22} - \frac{1}{2}g_{22}T)$$

Substituting $R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)$ and $T_{22} = Pg_{22}$:

$$r^2(a\ddot{a} + 2\dot{a}^2 + 2K) = 8\pi G(Pg_{22} - \frac{1}{2}g_{22}(3P - \rho))$$

Dividing by r^2 and g_{22}, and simplifying:

$$a\ddot{a} + 2\dot{a}^2 + 2K = 8\pi G(\frac{1}{2}P + \frac{1}{2}\rho)$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2} = 4\pi G(\rho + P)$$

Using the acceleration equation ($\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$), we get: **first Friedmann equation**:

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

Deriving the Fluid Equation

- The energy-momentum conservation equation is $\nabla_\mu T^{\mu\nu} = 0$.
- Considering the $\nu = 0$ component: $\nabla_\mu T^\mu_0 = 0$.

$$\nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{0\mu} T^\mu_\lambda$$

- After detailed calculation, this leads to the fluid equation:

$$\dot{\rho} = -3H(\rho + P)$$

This is a crucial equation describing how energy density changes with the expansion of the universe.

The Fundamental Equations of Cosmology

The two independent cosmological equations are:

1. **First Friedmann Equation:**

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

2. **Fluid Equation (Conservation of Energy):**

$$\dot{\rho} = -3H(\rho + P)$$

- We have 2 equations with 3 unknown functions: $a(t)$, $\rho(t)$, and $P(t)$.
- Therefore, we need an additional equation, an **equation of state**, relating ρ and P.

Relating Pressure and Density

- For the equation of state we use a simple relation: P=wρ where w is a constant.

Dominant Matter/Energy Types:

- At any given time period, a certain type of matter/energy can dominate the Universe's behavior.
- **Early Universe (Radiation Dominated):**
  - Extremely hot, kinetic energies of particles much larger than their rest mass ($E = \sqrt{p^2 + m^2} \approx p$).
  - These are ultra-relativistic or relativistic particles.
  - From statistical mechanics, a gas of relativistic particles has an energy density/pressure relation of $P = \frac{1}{3}\rho$.
  - So, for radiation, $w = \frac{1}{3}$.
- **Later Universe (Matter Dominated):**
  - As the universe expands, it cools, and kinetic energies become smaller.
  - Particles become non-relativistic ($E \approx m$).
  - For non-relativistic matter (or dust), $w = 0$.

How Density Changes with Expansion

Using $P = w\rho$ in the fluid equation $\dot{\rho} = -3H(\rho + P)$. For a flat Universe ($K = 0$), $H^2 = \frac{8\pi G}{3}\rho$ (consistent with WMAP observations).

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}\rho(1 + w)$$

Integrating this equation:

$$\int \frac{d\rho}{\rho} = -3(1 + w)\int \frac{da}{a}$$

$$\ln \rho = -3(1 + w)\ln a + C$$

$$\rho(t) = \rho_0 a(t)^{-3(1+w)}$$

where ρ_0 is the density at time t_0 when $a(t_0) = 1$.

- For **Radiation** ($w = \frac{1}{3}$):

$$\rho(t) = \rho_0 a(t)^{-4}$$

- For **Matter (Dust)** ($w = 0$):

$$\rho(t) = \rho_0 a(t)^{-3}$$

Cosmological Densities

From the first Friedmann equation ($H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$), we can define the **critical density** ρ_c for a flat universe ($K = 0$):

$$H^2 = \frac{8\pi G}{3}\rho_c \implies \rho_c = \frac{3H^2}{8\pi G}$$

Dividing the Friedmann equation by H^2:

$$1 + \frac{K}{a^2 H^2} = \frac{8\pi G}{3H^2}\rho$$

We can define **density parameters** $\Omega_i = \rho_i/\rho_c$:

$$1 = \Omega_r + \Omega_m + \Omega_k$$

Where Ω_r for radiation, Ω_m for matter, and $\Omega_k = -K/(a^2 H^2)$ for curvature. At t_0 (today, with $a(t_0) = 1$ and $H = H_0$):

$$1 = \frac{\rho_{r,0}}{\rho_{c,0}} + \frac{\rho_{m,0}}{\rho_{c,0}} + \frac{\rho_{\Lambda,0}}{\rho_{c,0}} = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$$

(Assuming $\Omega_{\Lambda,0}$ for dark energy, related to a constant $P = -\rho$).

Fate of the Universe and Critical Density

The first Friedmann equation:

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

We can write K in terms of current values:

$$K = \left(\frac{8\pi G}{3}\rho - H^2\right) a^2$$

- If $\frac{8\pi G\rho}{3} = H^2$ at any time, then $K = 0$.
- If $K = 0$ at any point in time, it will remain zero forever.
- This means if the density is exactly the critical density, the universe is flat and remains flat.
- If $\rho > \rho_c$, then $K > 0$ (closed universe).
- If $\rho < \rho_c$, then $K < 0$ (open universe).