Alexandria Quantum Computing Group (AleQCG)

INTERNATIONAL YEAR OF Quantum Science and Technology

# Day 3: Quantum Algorithms

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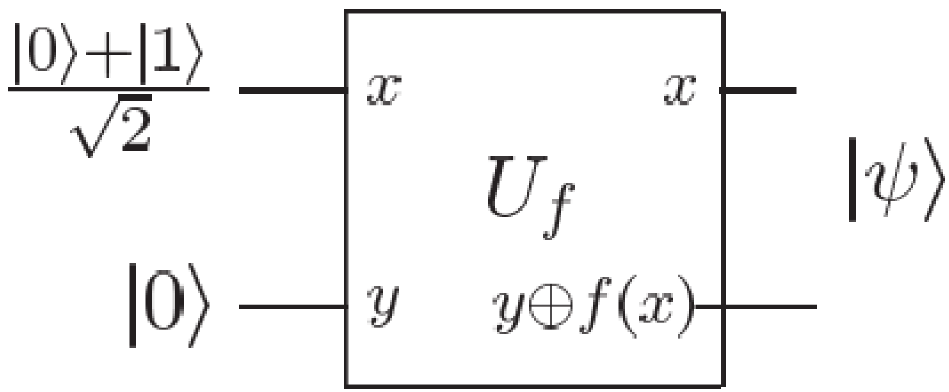
# Outline

- Quantum Parallelism
- Superposition Preparation
- Parallel Evaluation of a function
- Marking Solutions
- Deutsch-Jozsa Algorithm
- Grover’s Quantum Search Algorithm

# Quantum Parallelism

- *Quantum parallelism* is a fundamental feature of many quantum algorithms.
- Allows quantum computers to evaluate a function *f*(*x*) for many *different* values of *x* simultaneously.



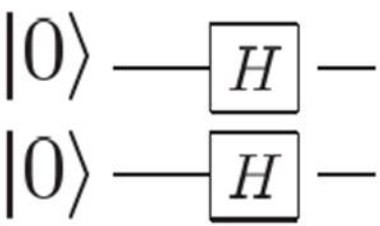

$$\frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}} .$$

# Quantum Parallelism

$$\overset{\text{input}}{\begin{Bmatrix} a_1\,|00\rangle \\ + \\ a_2\,|01\rangle \\ + \\ a_3\,|10\rangle \\ + \\ a_4\,|11\rangle \end{Bmatrix}} \xrightarrow{F} \begin{Bmatrix} a_1\,F\,|00\rangle \\ + \\ a_2\,F\,|01\rangle \\ + \\ a_3\,F\,|10\rangle \\ + \\ a_4\,F\,|11\rangle \end{Bmatrix} = \overset{\text{output}}{\begin{Bmatrix} b_1\,|00\rangle \\ + \\ b_2\,|01\rangle \\ + \\ b_3\,|10\rangle \\ + \\ b_4\,|11\rangle \end{Bmatrix}}$$

# Superposition Preparation

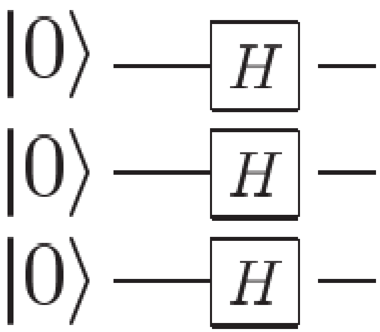
- This procedure can easily be generalized to functions on an arbitrary number of bits, by using a general operation known as the *Hadamard transform*, or sometimes the *Walsh– Hadamard transform*.
- *For n* = 2, $H^{\otimes 2}|00\rangle$




$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

# Superposition Preparation

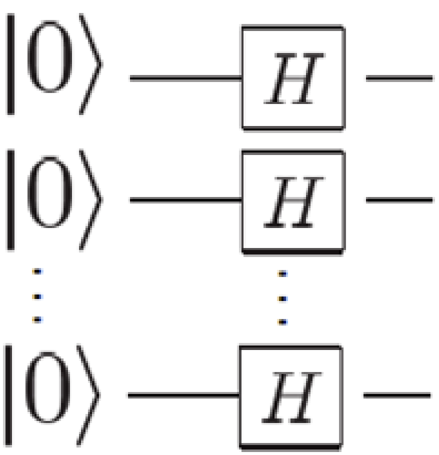
- *For n* = 3, $H^{\otimes 3}|000\rangle$



$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

$$=\frac{|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle}{2\sqrt{2}}$$

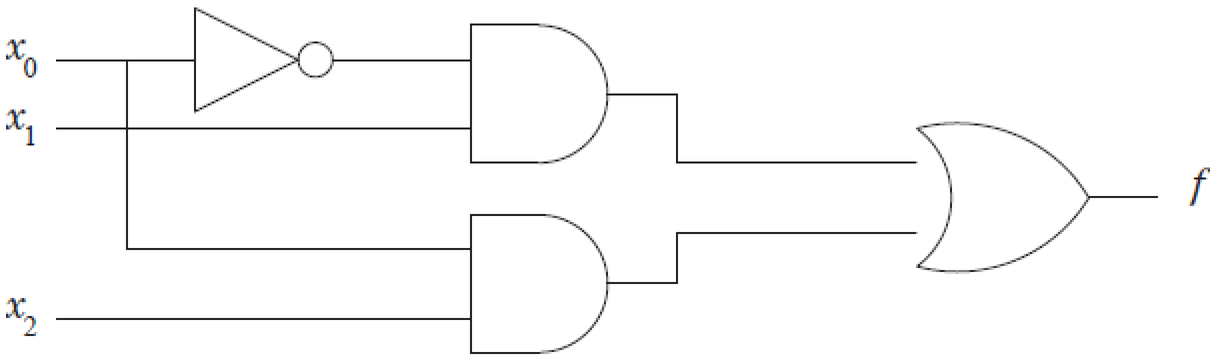
- For arbitrary n>0



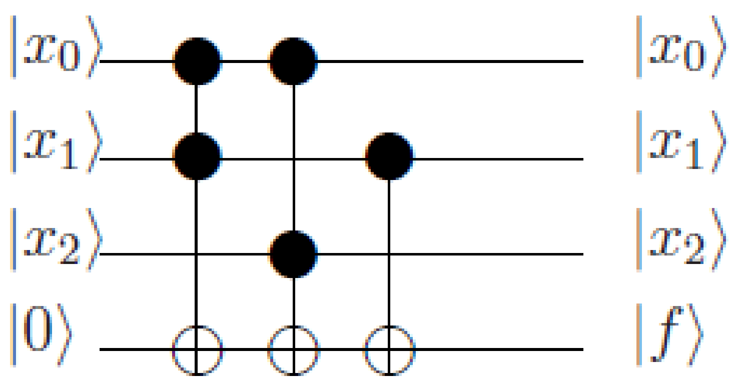
$$H^{\otimes n}|0\rangle^{\otimes n}=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

# Boolean Quantum Circuits

$$f = \overline{x_0}x_1 + x_0x_2$$



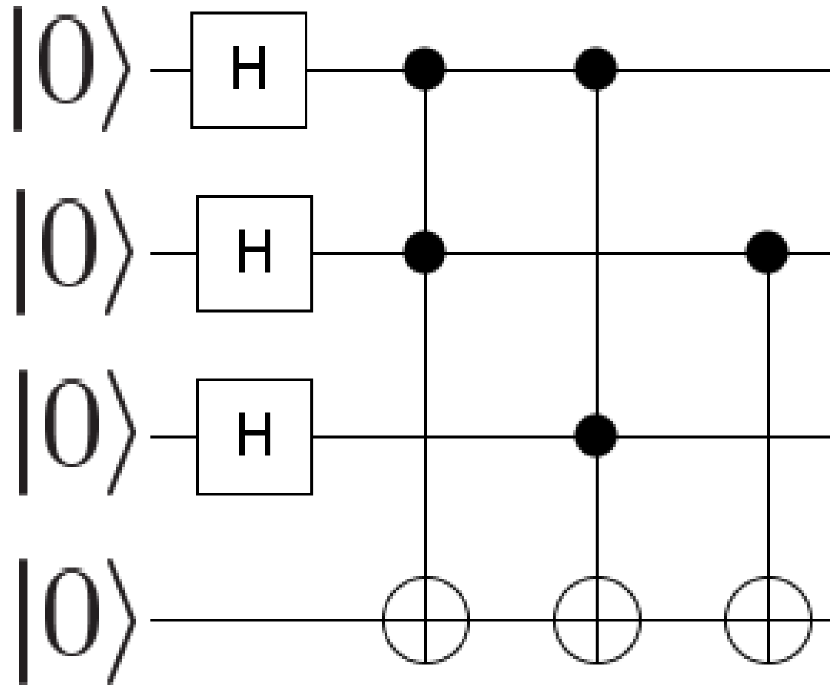

Digital circuit



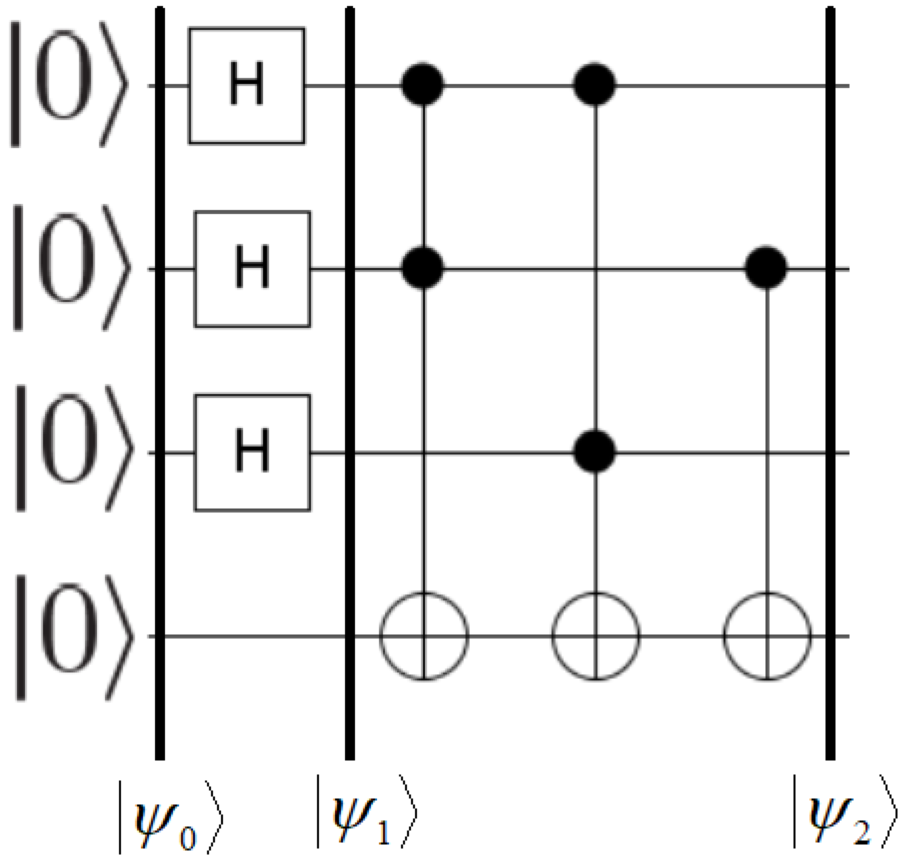

Quantum circuit

| $x_0$ | $x_1$ | $x_2$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Parallel Evaluation of $f$



# Tracing the Circuit



$$|\psi_0\rangle = |000\rangle \otimes |0\rangle$$

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

$$= \left(\frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2\sqrt{2}}\right) \otimes |0\rangle$$

$$= \left(\frac{|000,0\rangle + |001,0\rangle + |010,0\rangle + |011,0\rangle + |100,0\rangle + |101,0\rangle + |110,0\rangle + |111,0\rangle}{2\sqrt{2}}\right)$$

$$|\psi_2\rangle = \left(\frac{|000,0\rangle + |001,0\rangle + |010,1\rangle + |011,1\rangle + |100,0\rangle + |101,1\rangle + |110,0\rangle + |111,1\rangle}{2\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |100\rangle + |110\rangle\right) \otimes |0\rangle + \frac{1}{2\sqrt{2}}\left(|010\rangle + |011\rangle + |101\rangle + |111\rangle\right) \otimes |1\rangle$$

# Effect of Entanglement

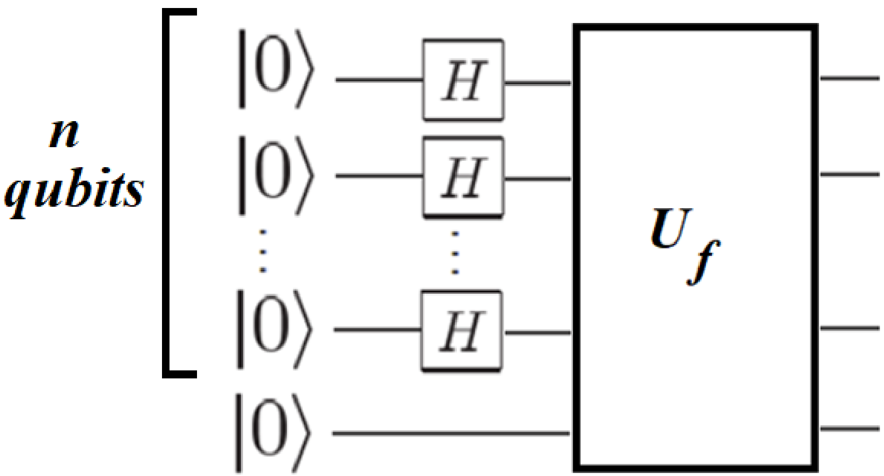
- If we measure the extra qubit and finds |0>, then the system collapses to

$$\frac{1}{2}\left(|000\rangle+|001\rangle+|100\rangle+|110\rangle\right)\otimes|0\rangle$$

- If we measure the extra qubit and finds |1>, then the system collapses to

$$\frac{1}{2}\left(|010\rangle+|011\rangle+|101\rangle+|111\rangle\right)\otimes|1\rangle$$

# General Form (Marking by Entanglement)




$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |0\rangle$$

$$|\psi_1\rangle = \left(H^{\otimes n} \otimes I\right)|\psi_0\rangle$$

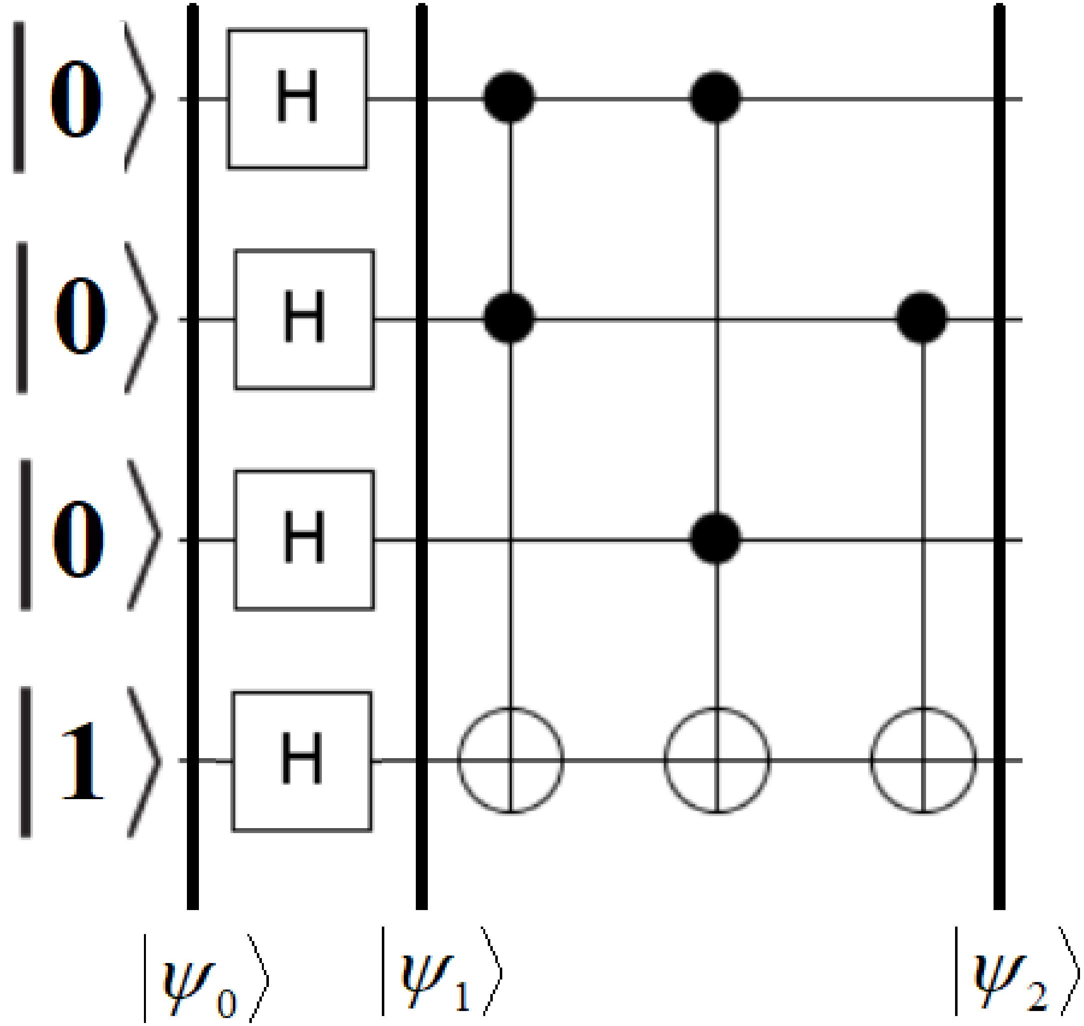
$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes |0\rangle$$

$$|\psi_2\rangle = U_f\,|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \left(|x\rangle \otimes |f(x)\rangle\right)$$

This method is called Marking the solutions by Entanglement

# Marking the Solutions by Phase Shift



$$\left|\psi_0\right\rangle = \left|000\right\rangle \otimes \left|1\right\rangle$$

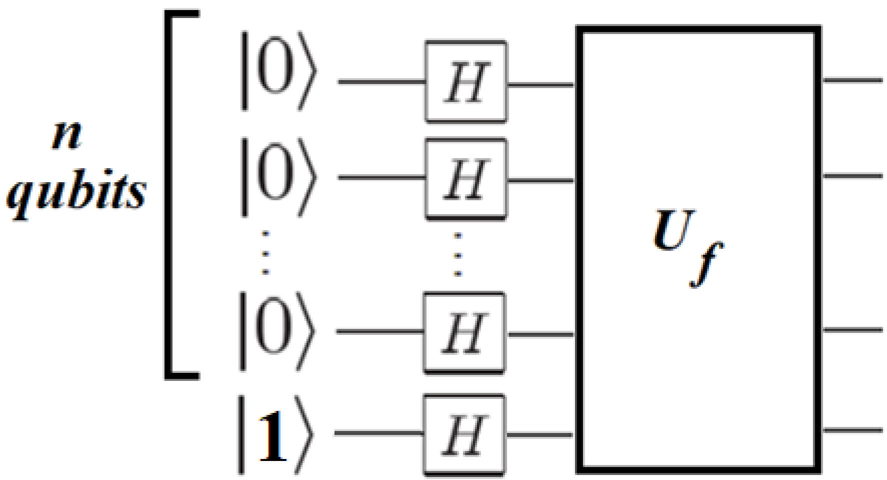
$$\left|\psi_1\right\rangle = \left(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}\right)\left(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}\right)\left(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}\right) \otimes \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}\right)$$

$$= \left(\frac{\left|000\right\rangle + \left|001\right\rangle + \left|010\right\rangle + \left|011\right\rangle + \left|100\right\rangle + \left|101\right\rangle + \left|110\right\rangle + \left|111\right\rangle}{2\sqrt{2}}\right) \otimes \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}\right)$$

$$\left|\psi_2\right\rangle = \left(\frac{\left|000\right\rangle + \left|001\right\rangle - \left|010\right\rangle - \left|011\right\rangle + \left|100\right\rangle - \left|101\right\rangle + \left|110\right\rangle - \left|111\right\rangle}{2\sqrt{2}}\right) \otimes \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{2\sqrt{2}}\left(\left|000\right\rangle + \left|001\right\rangle + \left|100\right\rangle + \left|110\right\rangle\right) - \frac{1}{2\sqrt{2}}\left(\left|010\right\rangle + \left|011\right\rangle + \left|101\right\rangle + \left|111\right\rangle\right)\right) \otimes \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}\right)$$

# General Form (Marking by Phase Shift)




$$|\psi_0\rangle = |0\rangle^{\otimes n}$$

$$|\psi_1\rangle = H^{\otimes n}|\psi_0\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

$$|\psi_2\rangle = U_f|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle$$

# Deutsch-Jozsa Algorithm

# Deutsch-Jozsa Algorithm

Problem: Given an unknown function $f(x)$ which is promised to be one of two kinds :-

1. $f(x)$ is constant $\forall x$, $f(x)=0$ or $f(x)=1$
2. $f(x)$ is balanced, i.e. equals to 0 for half inputs and equals to 1 for half inputs.

In classical case, we need $\frac{2^n}{2}+1$ tests.

In quantum case, using $U_f$ to calculate $f(x)$, we need only one test using quantum parallelism and quantum interference.

# Example

| $x_0$ | $x_1$ | $x_2$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

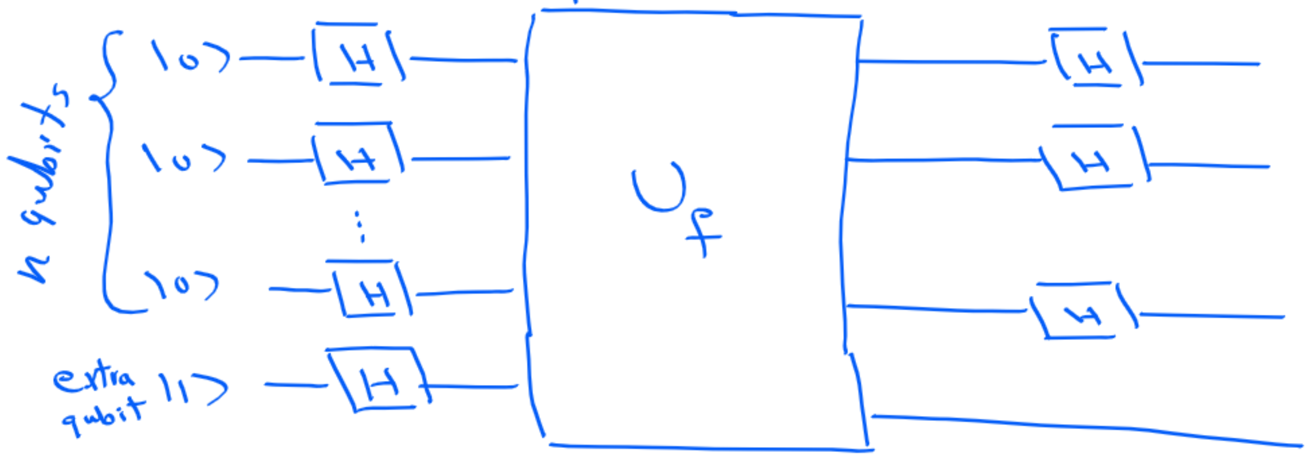
Constant Function

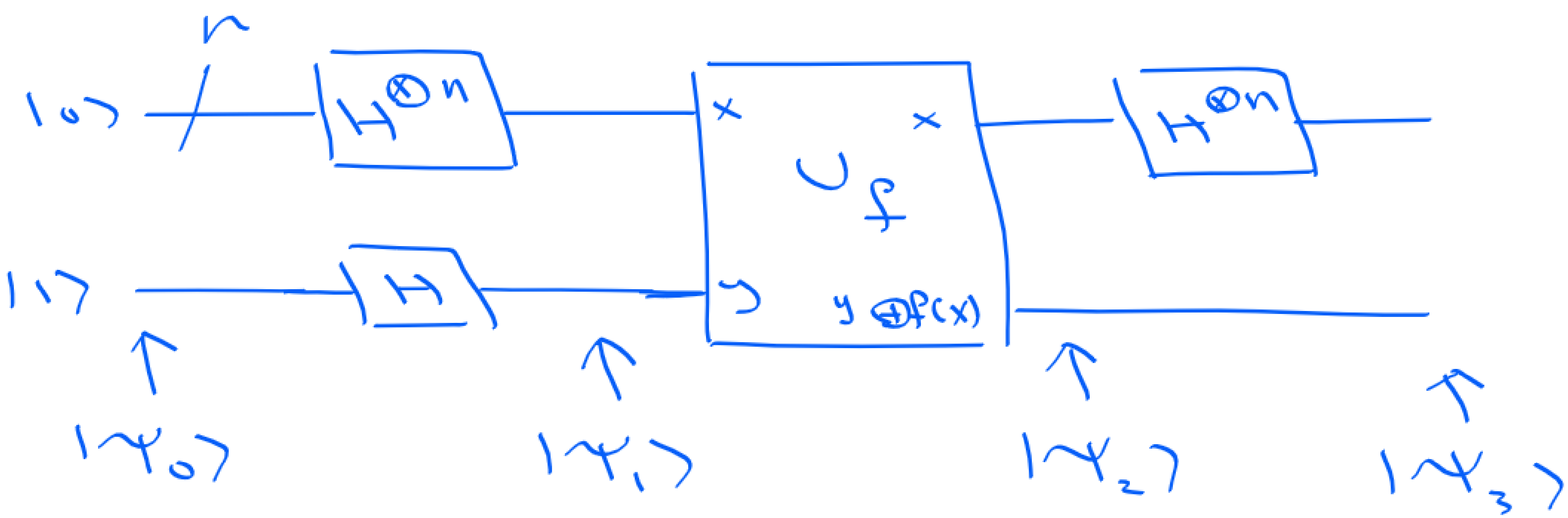
| $x_0$ | $x_1$ | $x_2$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

Balanced Function

# The Quantum Circuit

Let f(x) have n inputs



## Tracing the Algorithm

$$|\psi_0\rangle = |0\rangle^{\otimes n} \otimes |1\rangle$$

$$|\psi_1\rangle = H^{\otimes n+1} |\psi_0\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{N} |x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = U_f |\psi_1\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} |x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

(marking by phase shift, ignore the extra qubit)

N.B.

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{z \in \{0,1\}} (-1)^{x \cdot z} |z\rangle$$

$\underline{x = 0}$

$$\left.\begin{array}{l} \underline{z=0}, \ (-1)^{0 \cdot 0}|0\rangle = |0\rangle \\ \underline{z=1}, \ (-1)^{0 \cdot 1}|1\rangle = |1\rangle \end{array}\right\} \ (|0\rangle + |1\rangle)$$

$\underline{x = 1}$

$$\left.\begin{array}{l} \underline{z=0}, \ (-1)^{1 \cdot 0}|0\rangle = |0\rangle \\ \underline{z=1}, \ (-1)^{1 \cdot 1}|1\rangle = -|1\rangle \end{array}\right\} \ (|0\rangle - |1\rangle)$$

$$|x_1\rangle \text{ — } \boxed{H} \text{ — }$$
$$|x_2\rangle \text{ — } \boxed{H} \text{ — }$$

$$H^{\otimes 2}(|x_1\rangle \otimes |x_2\rangle)$$

$$= \frac{1}{\sqrt{2}} \sum_{z_1} (-1)^{x_1 z_1} |z_1\rangle \cdot \frac{1}{\sqrt{2}} \sum_{z_2} (-1)^{x_2 z_2} |z_2\rangle$$

$$= \frac{1}{\sqrt{2^2}} \sum_{z_1, z_2} (-1)^{x_1 z_1 + x_2 z_2} |z_1, z_2\rangle$$

In general,

$$H^{\otimes n} |x_1 x_2 \cdots x_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{z_1, \ldots, z_n} (-1)^{x_1 z_1 + x_2 z_2 + \cdots + x_n z_n} |z_1, z_2, \ldots, z_n\rangle$$

→ inner product

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z} (-1)^{x \cdot z} |z\rangle$$

$$|x\rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Ex:

$$H^{\otimes 3}|101\rangle = \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2^3}}\Big[|000\rangle - |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle - |110\rangle + |111\rangle\Big]$$

| $\lvert 101\rangle$ | inner | sign | |
|---|---|---|---|
| | 0 | + | $\lvert 000\rangle$ |
| | 1 | − | $\lvert 001\rangle$ |
| | 0 | + | $\lvert 010\rangle$ |
| | 1 | − | $\lvert 011\rangle$ |
| | 1 | − | $\lvert 100\rangle$ |
| | 2 | + | $\lvert 101\rangle$ |
| | 1 | − | $\lvert 110\rangle$ |
| | 2 | + | $\lvert 111\rangle$ |

Now to calculate $|\psi_3\rangle$

$$|\psi_3\rangle = (H^{\otimes n} \otimes I)|\psi_2\rangle = (H^{\otimes n} \otimes I)\frac{1}{\sqrt{2^n}}\sum_x (-1)^{f(x)}|x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2^n}}\sum_x (-1)^{f(x)} \underbrace{H^{\otimes n}|x\rangle}_{\downarrow} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2^n}}\sum_z (-1)^{xz}|z\rangle$$

$$= \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{z=0}^{2^n-1}(-1)^{f(x)+xz}|z\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_3\rangle = \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{z=0}^{2^n-1}(-1)^{f(x)+xz}\,|z\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$|z\rangle = |00\cdots0\rangle$ $\qquad\qquad$ $|11\cdots1\rangle$

$$\text{if } |z\rangle = |0\rangle, \quad \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^{f(x)}$$

$$\text{if } f(x)=0, \quad \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^0 = 1$$

$$\text{if } f(x)=1, \quad \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^1 = -1$$

Apply measurment on the first n qubits

If $f(x)$ is constant

$f(x)=0$ $\downarrow$ $\sum_x \frac{1}{2^n} = 1$

$f(x)=1$ $\downarrow$ $-\sum \frac{1}{2^n} = -1$

Amplitude of $|0\rangle^{\otimes n}$ is $+1$ or $-1$ depends on $f(x)$.

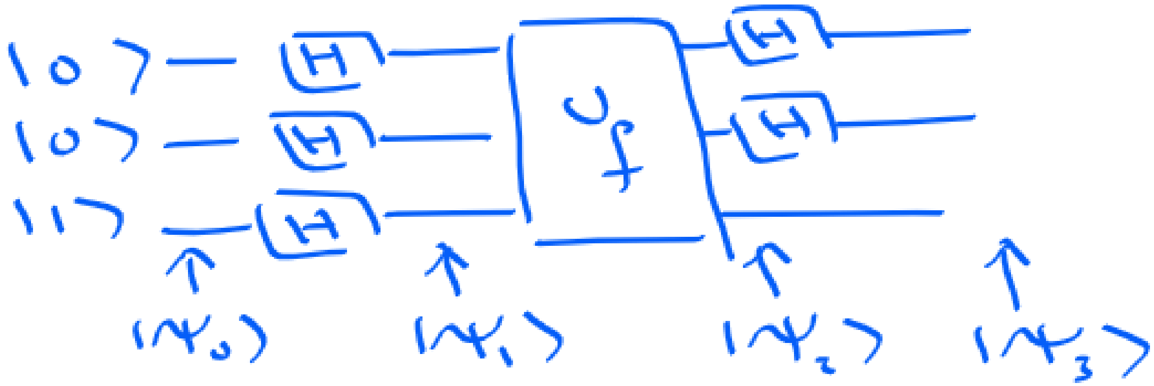
Because $|\psi_3\rangle$ is of unit length then it follows that all other amplitude must be zero.

If $f(x)$ is balanced

The positive and negative contributions to amplitude of $|0\rangle^{\otimes n}$ cancel each other, leaving amplitude zero, and the measure must yield a state other than $|0\rangle^{\otimes n}$.

Ex: $f(x) = 1$, $n = 2$ (constant)

| | | |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |




(Ignore the extra qubit)

$$|\psi_0\rangle = |00\rangle$$

$$|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{2}\left(-|00\rangle - |01\rangle - |10\rangle - |11\rangle\right)$$

$$= \frac{-1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|\psi_2\rangle = \frac{-1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|\psi_3\rangle = -\frac{1}{2}\Big[\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$+\frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

$$+\frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

$$+\frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)\Big]$$

$$= -\frac{1}{2}\left(2|00\rangle\right) = -|00\rangle$$

| $x_0$ | $x_1$ | $f$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

<u>Ex:</u> $f(x_0, x_1) = x_0 \oplus x_1$, $n=2$ (balanced)

$$|\psi_0\rangle = |00\rangle$$

$$|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

$$\begin{aligned}
|\psi_3\rangle = \frac{1}{2}\Big[ & \frac{1}{2}\left(\overset{+}{|00\rangle} \overset{+}{+} |01\rangle \overset{+}{+} |10\rangle \overset{+}{+} |11\rangle\right) \\
& -\frac{1}{2}\left(\overset{-}{|00\rangle} \overset{+}{-} |01\rangle \overset{-}{+} |10\rangle \overset{+}{-} |11\rangle\right) \\
& -\frac{1}{2}\left(\overset{-}{|00\rangle} \overset{-}{+} |01\rangle \overset{+}{-} |10\rangle \overset{+}{-} |11\rangle\right) \\
& +\frac{1}{2}\left(\overset{+}{|00\rangle} \overset{-}{-} |01\rangle \overset{-}{-} |10\rangle \overset{+}{+} |11\rangle\right) \Big]
\end{aligned}$$

$$= \frac{1}{2}\left(2|11\rangle\right) = |11\rangle \neq |00\rangle$$

# Grover's Quantum Search Algorithm

# Unstructured Search Problem

- Consider an unstructured list $L$ of $N$ items.
- For simplicity and without loss of generality we will assume that $N = 2^n$ for some positive integer $n$.
- Suppose the items in the list are labelled with the integers {0, 1, ..., $N-1$}, and consider a function (oracle) $f$ which maps an item $i \in L$ to either 0 or 1 according to some properties this item should satisfy, i.e. $f : L \rightarrow \{0, 1\}$.
- The problem is to find any $i \in L$ such that $f(i) = 1$ assuming that such $i$ exists in the list.

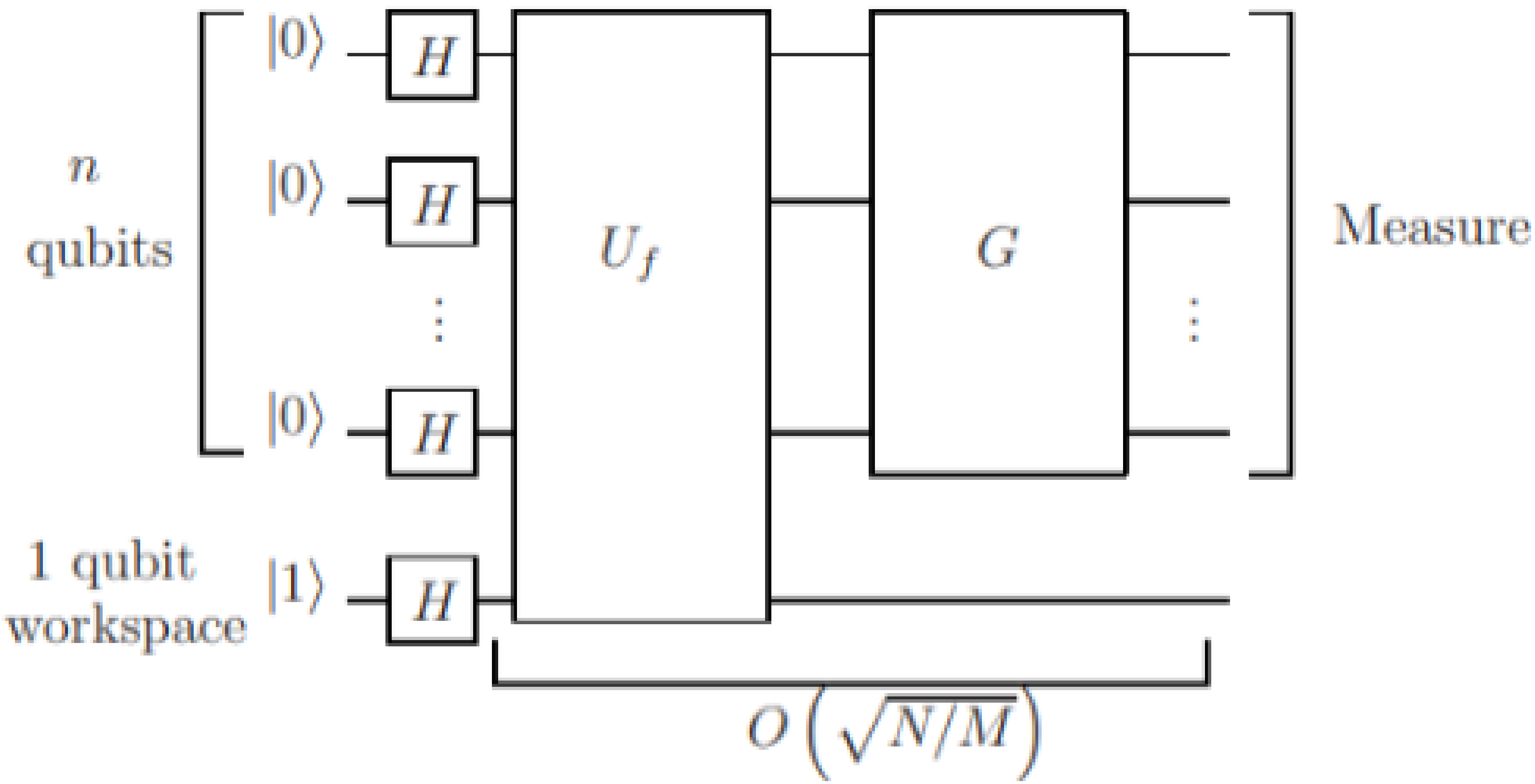
# Classical Searching

- In conventional computers, solving this problem needs

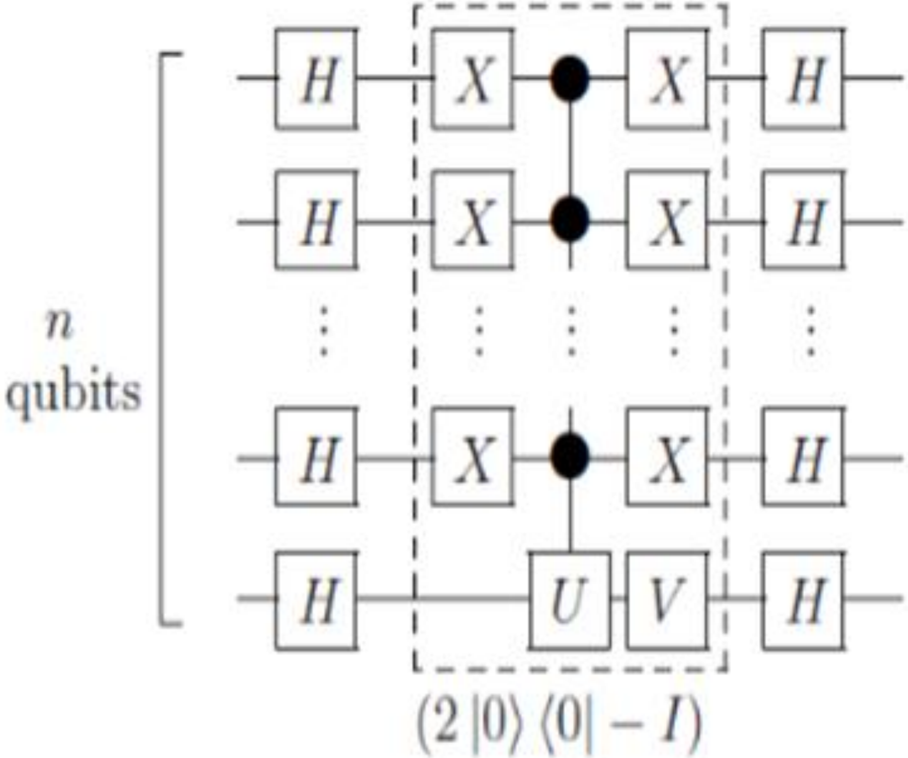
O($N/M$) calls to the oracle (query), where $M$ is the number of items that satisfy the oracle.

- The unstructured search problem can be considered as a general domain for a wide range of applications in computer science, for example:
  - The **database searching problem**, where we are looking for an item in an unsorted list.
  - The **Boolean satisfiability problem**, where we have a Boolean expression with $n$ Boolean variables and we are looking for any variable assignment that satisfies this expression.

# Quantum Circuit for Grover's algorithm



# Grover's Diffusion Operator G



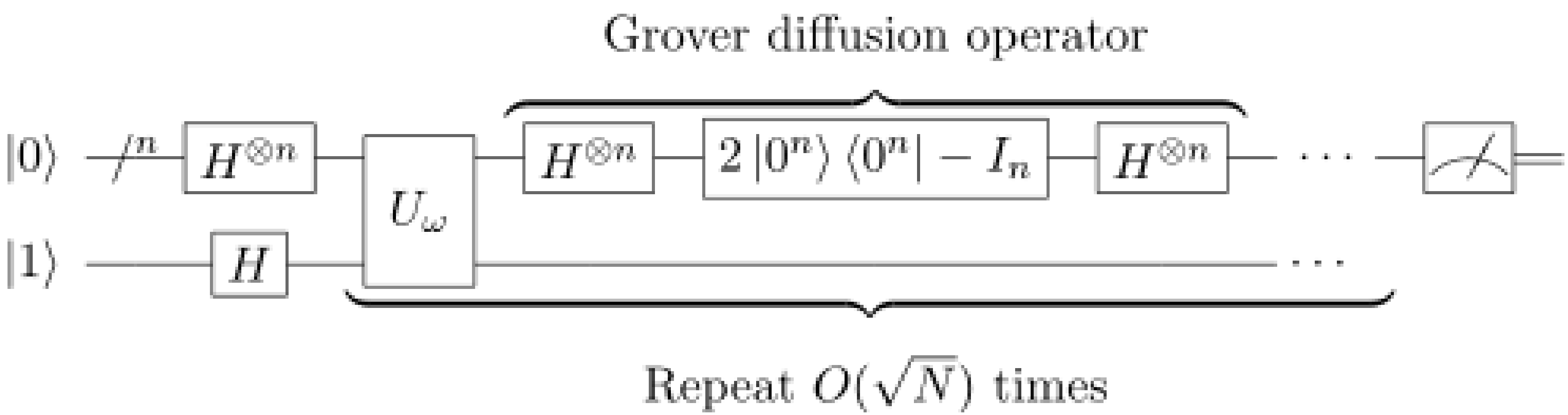

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

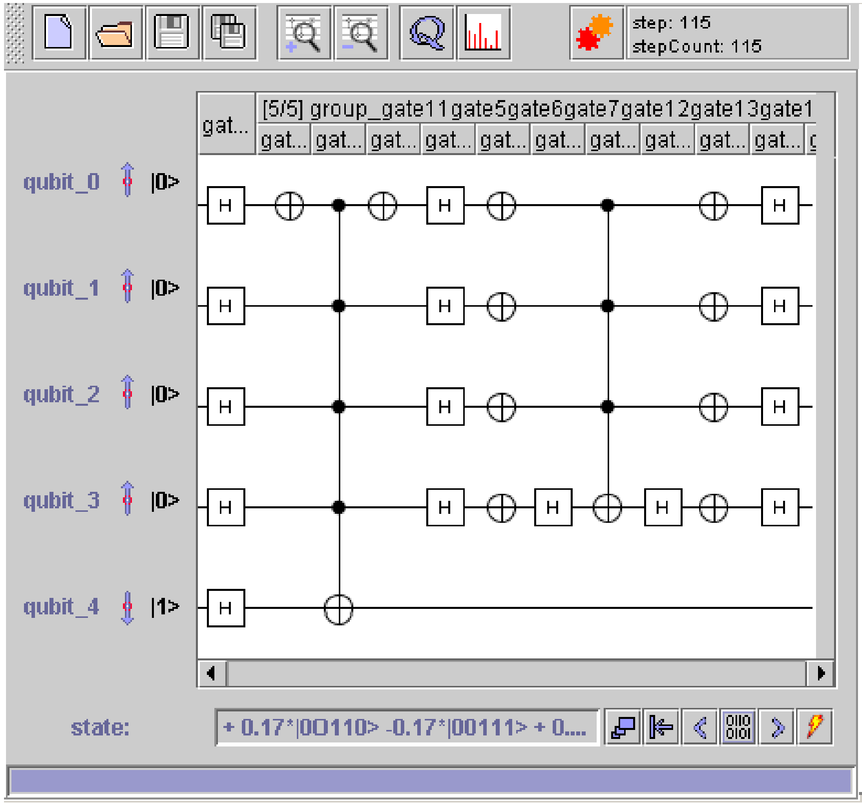
Quantum circuit for the diffusion operator $G$ over $n$ qubits.

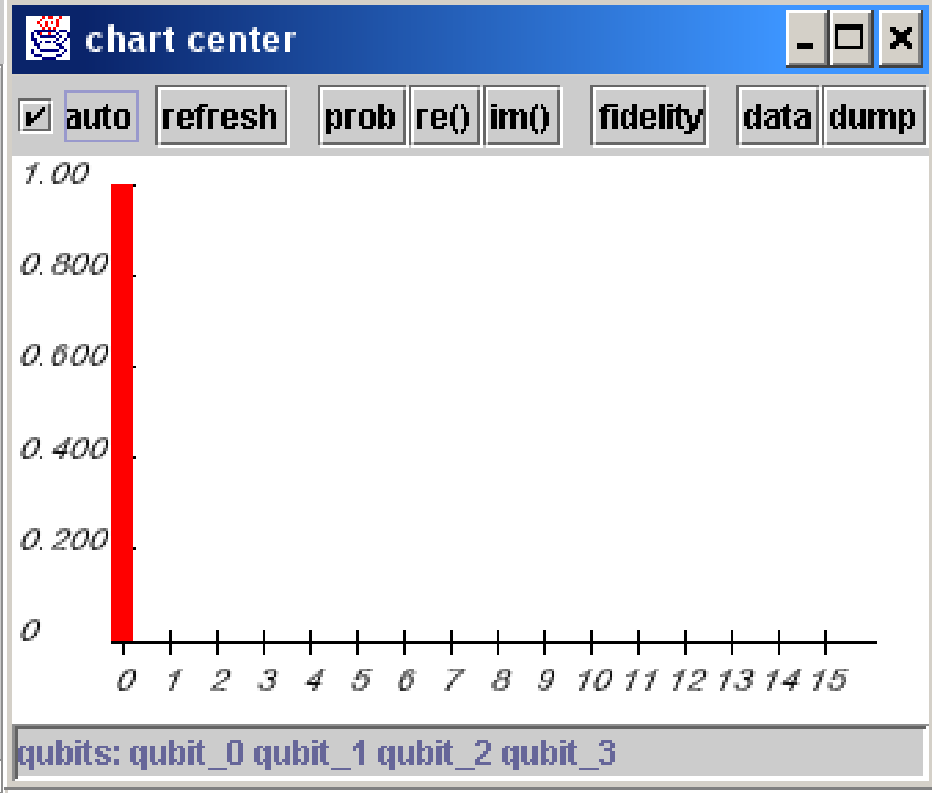
# Steps

1. Prepare a quantum register of $n + 1$ qubits. The first $n$ qubits all in state |0> and the extra qubit in state |1>.
2. Apply the Hadamard gate $H$ on each of the $n + 1$ qubits in parallel.
3. Iterate the following steps $q$ times,
   i. Apply the oracle $U_f$.
   ii. Apply the diffusion operator $G$ on the first $n$ qubits.
4. Measure the first $n$ qubits to get the result with probability $P_s$

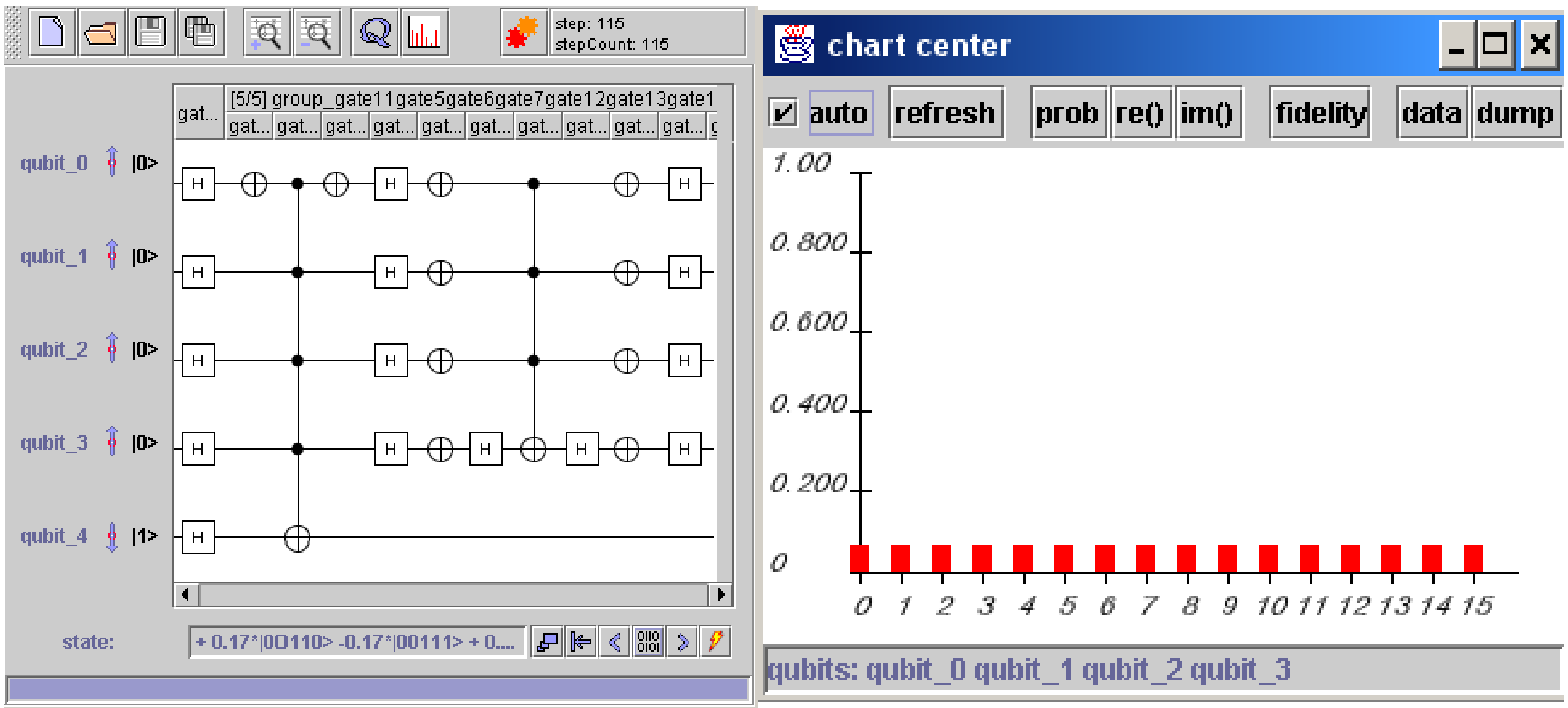


# Example: Search for ?.



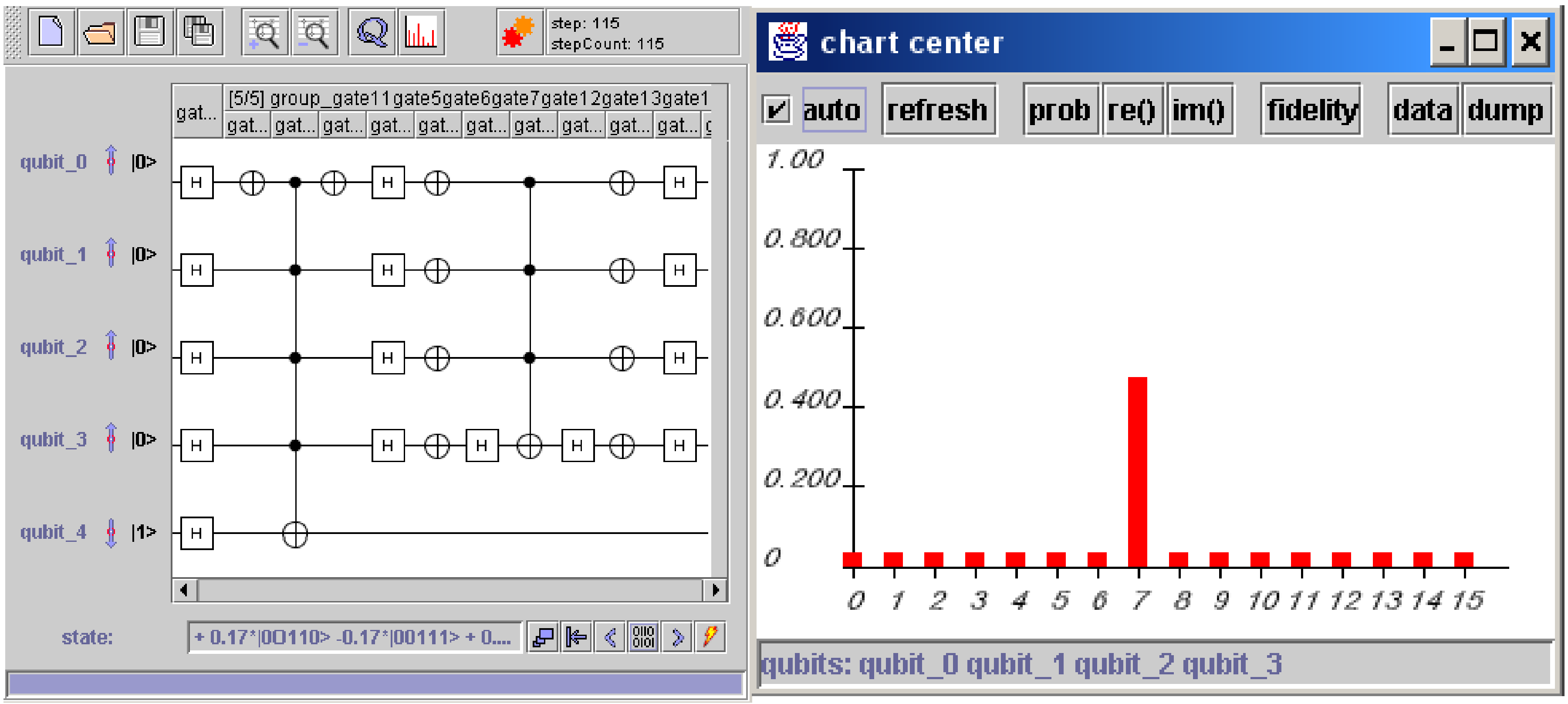



# Example: Search for ?.



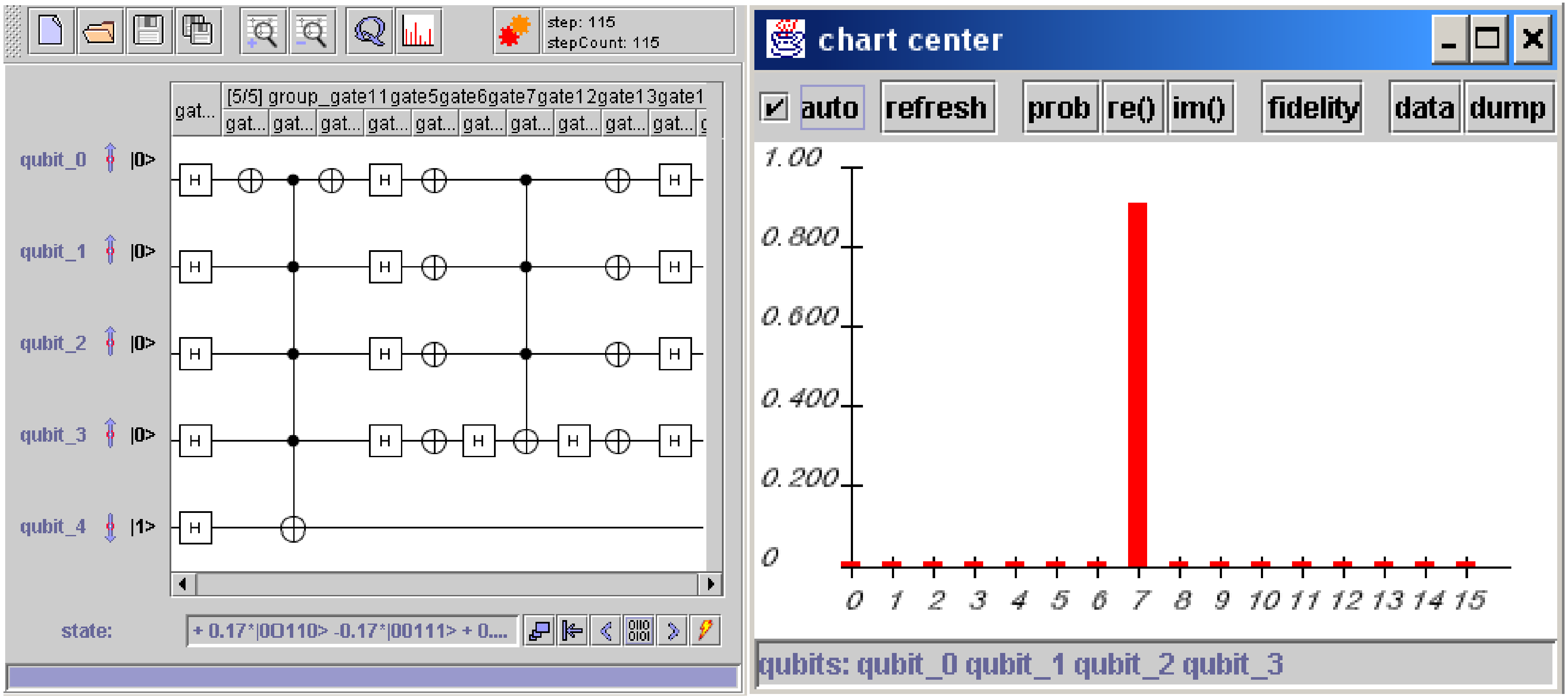

initialization

# Example: Search for 7.



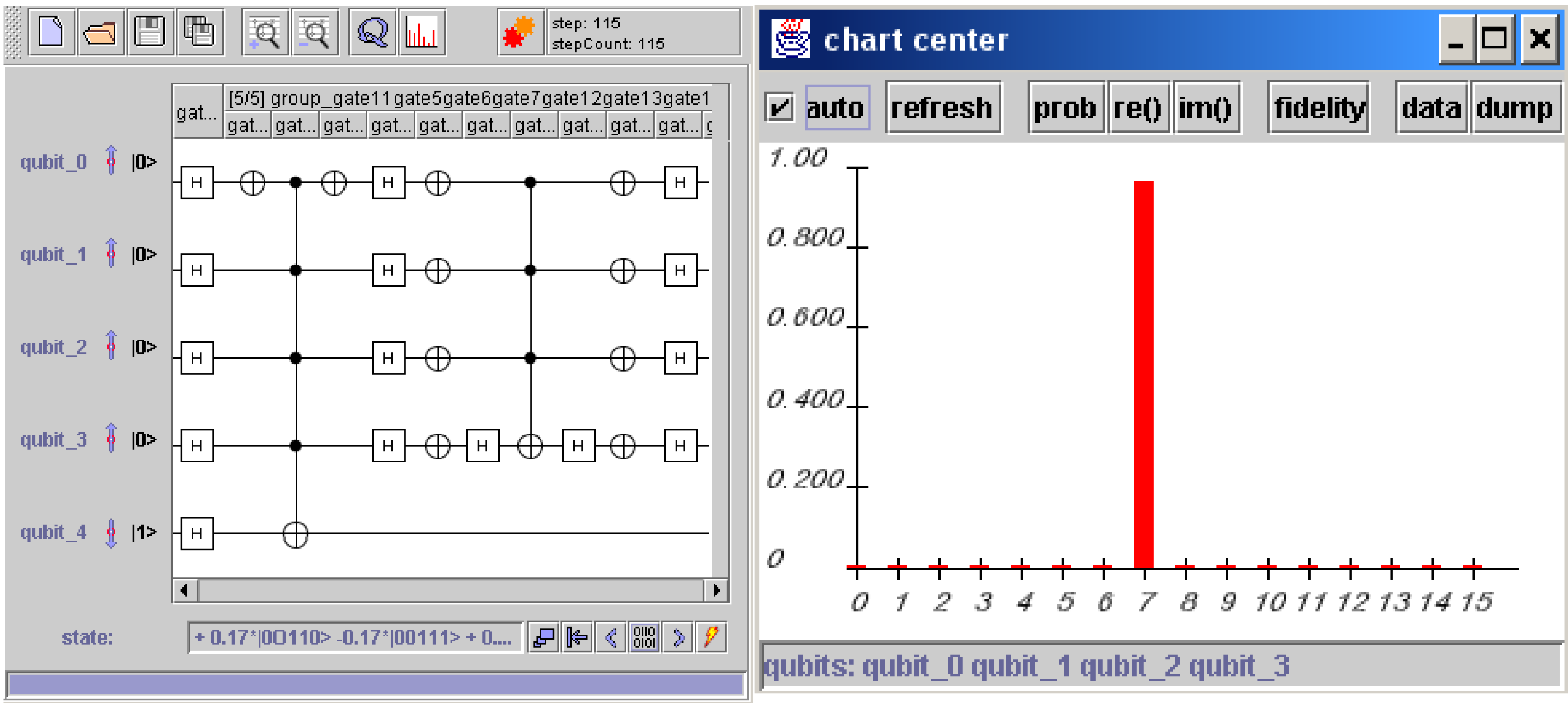

1st Iteration

# Example: Search for 7.



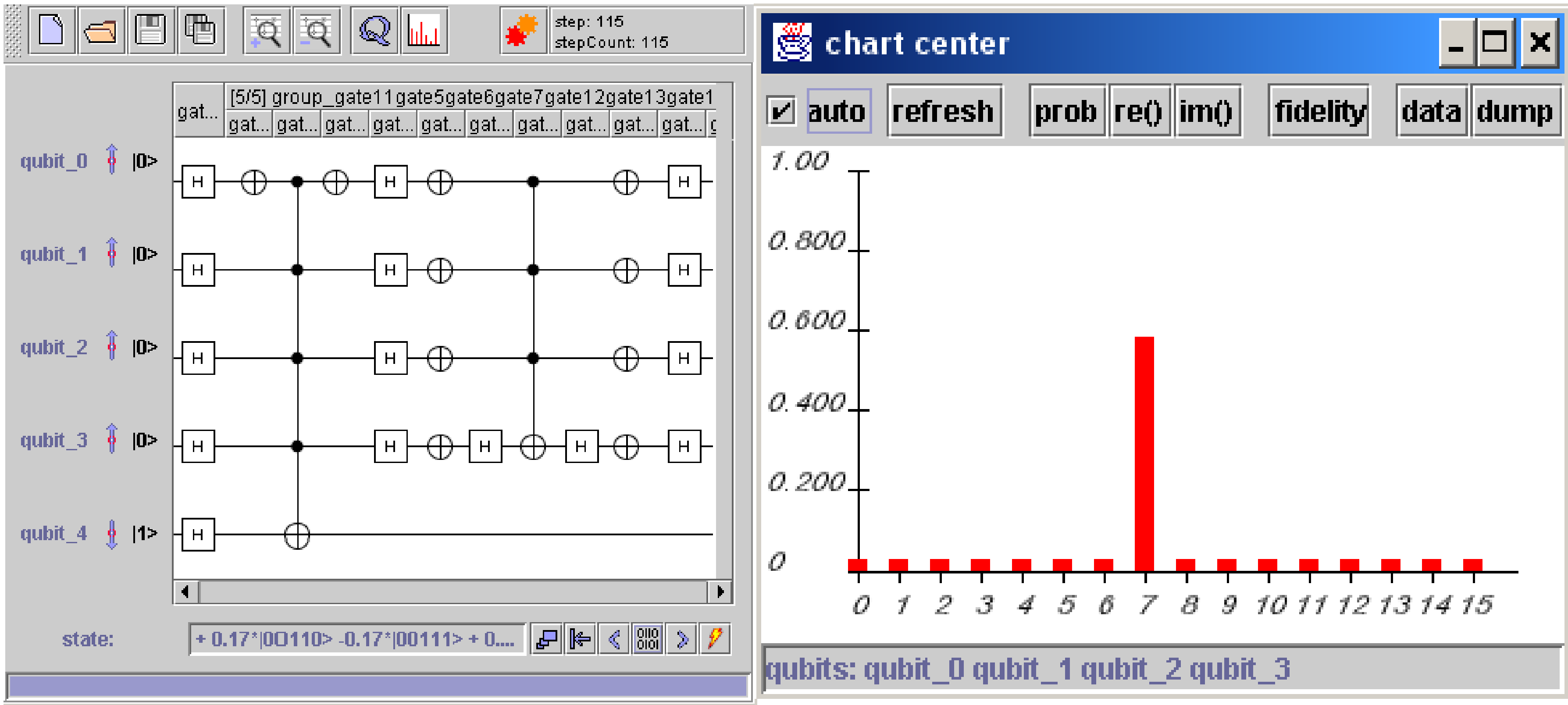

2nd Iteration

# Example: Search for 7.



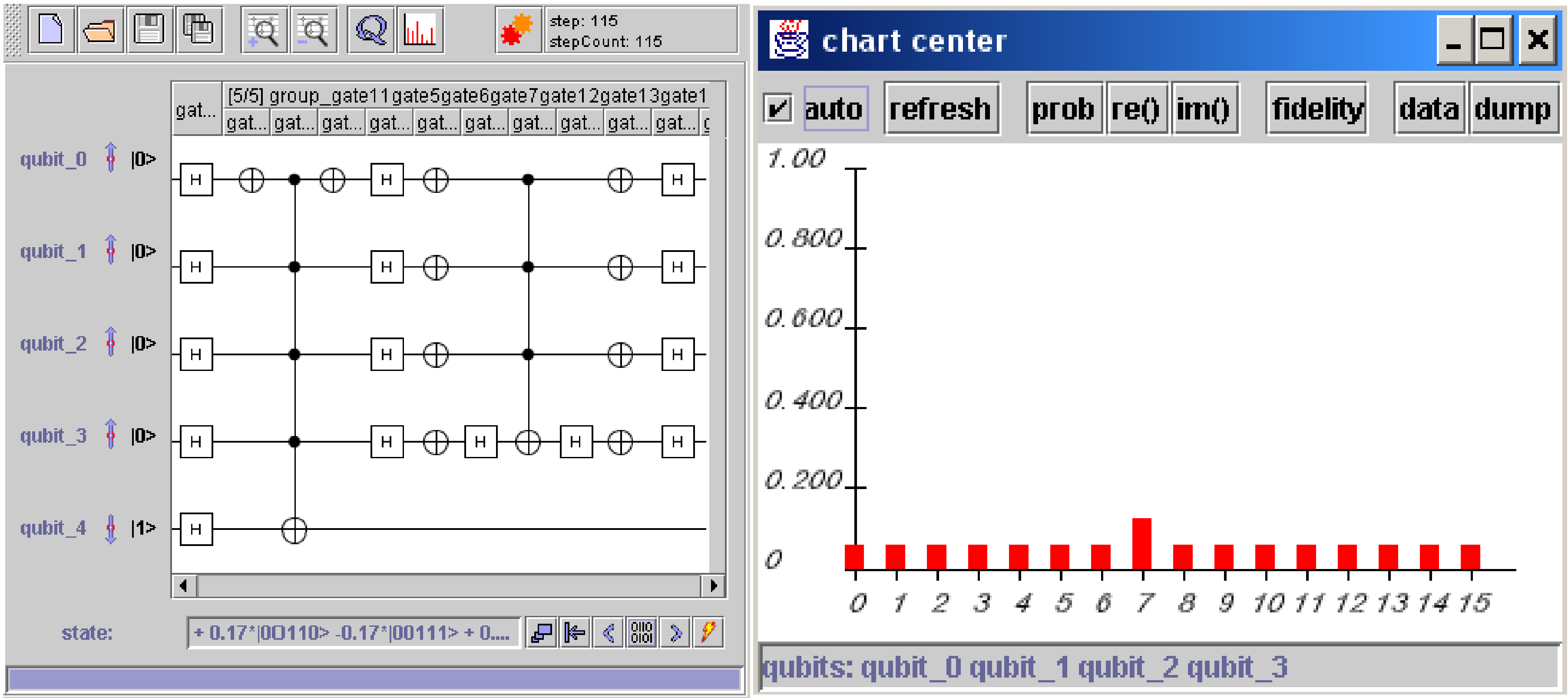

3rd Iteration

# Example: Search for 7.




4th Iteration

# Example: Search for 7.




$5^{th}$ Iteration

$$\left(\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\boxed{\frac{1}{\sqrt{2^3}}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}}\right)$$

**Initialization**

$$\Downarrow U_f$$

$$\left(\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}},\boxed{\frac{-1}{\sqrt{2^3}}},\frac{1}{\sqrt{2^3}},\frac{1}{\sqrt{2^3}}\right)$$

**Marking the Solution**

$$\Downarrow G$$

$$\alpha_j' = 2\langle\alpha\rangle - \alpha_j$$

$$\langle\alpha\rangle = \frac{1}{2^3}\left(7*\frac{1}{\sqrt{2^3}}+1*\frac{-1}{\sqrt{2^3}}\right)=\frac{6}{2^3\sqrt{2^3}}=\frac{3}{4\sqrt{2^3}}$$

$$G\frac{1}{\sqrt{2^3}}\rightarrow 2\left(\frac{3}{4\sqrt{2^3}}\right)-\left(\frac{1}{\sqrt{2^3}}\right)=\frac{1}{2\sqrt{2^3}}$$

**Amplitude Amplification**

$$G\frac{-1}{\sqrt{2^3}}\rightarrow 2\left(\frac{3}{4\sqrt{2^3}}\right)-\left(\frac{-1}{\sqrt{2^3}}\right)=\frac{5}{2\sqrt{2^3}}$$

$$\left(\frac{1}{2\sqrt{2^3}},\frac{1}{2\sqrt{2^3}},\frac{1}{2\sqrt{2^3}},\frac{1}{2\sqrt{2^3}},\frac{1}{2\sqrt{2^3}},\boxed{\frac{5}{2\sqrt{2^3}}},\frac{1}{2\sqrt{2^3}},\frac{1}{2\sqrt{2^3}}\right)$$

# Inversion about the Mean

Apply the *Diffusion Operator* $G$ on the first $n$ qubits. The diagonal representation of $G$ can take this form

$$G = H^{\otimes n} \left(2 \left|0\right\rangle \left\langle 0\right| - I_n\right) H^{\otimes n},$$

where the vector $\left|0\right\rangle$ used is of length $N = 2^n$, and $I_n$ is the identity matrix of size $2^n \times 2^n$. Consider a general system $\left|\psi\right\rangle$ of $n$-qubit quantum register:

$$\left|\psi\right\rangle = \sum_{j=0}^{N-1} \alpha_j \left|j\right\rangle.$$

The effect of applying $G$ on $\left|\psi\right\rangle$ produces,

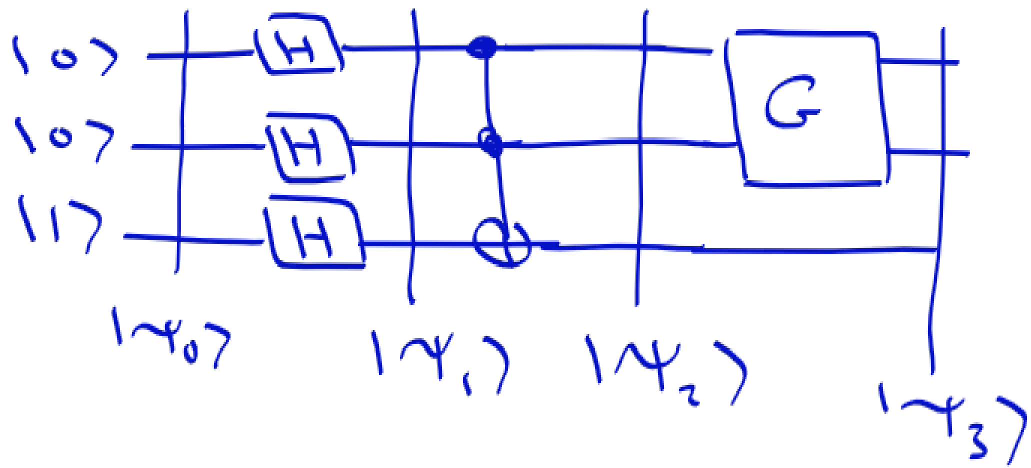
$$G\left|\psi\right\rangle = \sum_{j=0}^{N-1} \left[-\alpha_j + 2\left\langle\alpha\right\rangle\right] \left|j\right\rangle,$$

where, $\left\langle\alpha\right\rangle = \frac{1}{N}\sum_{j=0}^{N-1} \alpha_j$ is the mean of the amplitudes of the states in the superposition, i.e. each amplitude $\alpha_j$ will be transformed according to the following relation:

$$\alpha_j \to \left[-\alpha_j + 2\left\langle\alpha\right\rangle\right].$$

Ex

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |




$$|\psi_0\rangle = |00\rangle$$

$$|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

$$G : \alpha_j \longrightarrow 2\langle\alpha\rangle - \alpha_j$$

$$\langle\alpha\rangle = \frac{1}{4}\left(3\left(\frac{1}{2}\right) + 1\left(-\frac{1}{2}\right)\right) = \frac{1}{4}$$

$$\frac{1}{2} : 2\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right) = 0$$

$$-\frac{1}{2} : 2\left(\frac{1}{4}\right) - \left(-\frac{1}{2}\right) = 1$$

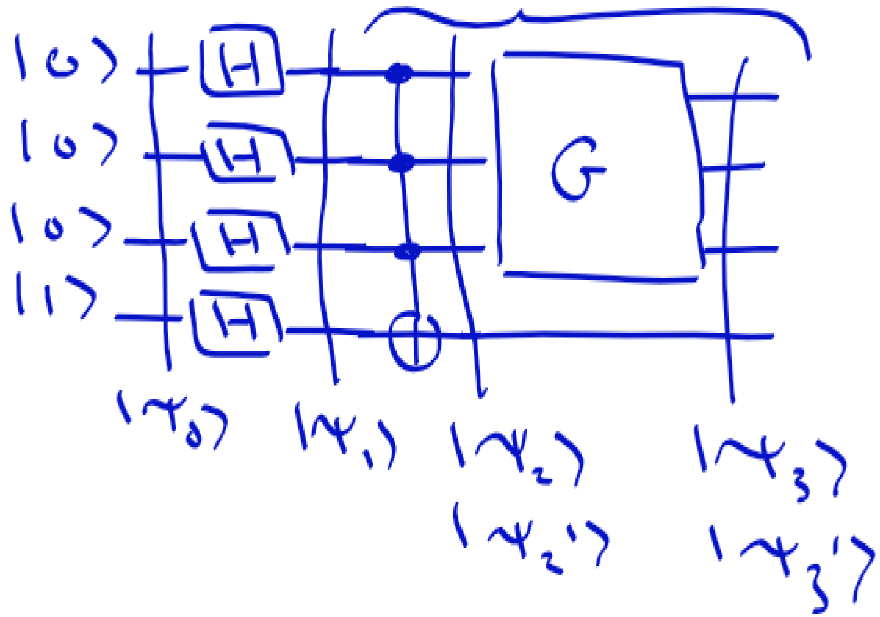
$|100\rangle$ $\quad$ $|101\rangle$ $\quad$ $|110\rangle$ $\quad$ $|111\rangle$

$\frac{1}{2} \to 0$ $\quad$ $\frac{1}{2} \to 0$ $\quad$ $\frac{1}{2} \to 0$ $\quad$ $-\frac{1}{2} \to 1$

$$|\psi_3\rangle = |111\rangle$$

Hint: If the number of solutions is 25% of the search space, Grover's Algorithm will terminate after 1 iteration with prob. of suc. 100%.

Ex:




| | |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| ⋮ | ⋮ |
| 1 1 0 | 0 |
| 1 1 1 | 1 |

$$|\psi_0\rangle = |000\rangle$$

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + \cdots + |110\rangle + |111\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{2\sqrt{2}}\Big(\underbrace{|000\rangle + |001\rangle + \cdots + |110\rangle}_{7\left(\frac{1}{2\sqrt{2}}\right)} \underbrace{- |111\rangle}_{1\left(\frac{-1}{2\sqrt{2}}\right)}\Big)$$

$$\langle \alpha \rangle = \frac{1}{8}\left(7\left(\frac{1}{2\sqrt{2}}\right) + 1\left(\frac{-1}{2\sqrt{2}}\right)\right) = \frac{3}{8\sqrt{2}}$$

$$\alpha_j \to 2\langle \alpha \rangle - \alpha_j$$

$$\alpha_j \begin{cases} \frac{1}{2\sqrt{2}} : & 2\left(\frac{3}{8\sqrt{2}}\right) - \left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{4\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} : & 2\left(\frac{3}{8\sqrt{2}}\right) - \left(\frac{-1}{2\sqrt{2}}\right) = \boxed{\frac{5}{4\sqrt{2}}} \end{cases}$$

prob. 0.03125

prob. 0.78125

$$|\psi_3\rangle = \frac{1}{4\sqrt{2}}\left(|000\rangle + |001\rangle + \cdots + |110\rangle\right) + \frac{5}{4\sqrt{2}}|111\rangle$$

2nd iteration

Apply $U_f$ : $|\psi_1'\rangle = \frac{1}{4\sqrt{2}}\left(|000\rangle + \cdots + |110\rangle\right) - \frac{5}{4\sqrt{2}}|111\rangle$

$$\langle \alpha \rangle = \frac{1}{8}\left(7\left(\frac{1}{4\sqrt{2}}\right) + 1\left(\frac{-5}{4\sqrt{2}}\right)\right)$$

$$= \frac{1}{16\sqrt{2}}$$

$$\alpha_j \begin{cases} \dfrac{1}{4\sqrt{2}} \rightarrow 2\left(\dfrac{1}{16\sqrt{2}}\right) - \left(\dfrac{1}{4\sqrt{2}}\right) = -\dfrac{1}{8\sqrt{2}} & 0.0078 \\ \dfrac{-5}{4\sqrt{2}} \rightarrow 2\left(\dfrac{1}{16\sqrt{2}}\right) - \left(\dfrac{-5}{4\sqrt{2}}\right) = \dfrac{11}{8\sqrt{2}} & 0.9455 \end{cases}$$

$$|\psi_3'\rangle = \frac{-1}{8\sqrt{2}}\left(|000\rangle + \cdots + |110\rangle\right) + \frac{11}{8\sqrt{2}}|111\rangle$$

we should stop and apply measurment after $\left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor$

In this Ex: $\lfloor 2.22 \rfloor \rightarrow 2$ iterations

Thank you