PT 616

Selected Topics in Theoretical Physics

Spring 2021

Introduction to Theory of Superconductivity

Instructor: Abdelhamid Galal

Text: Michael Tinkham, Introduction to Superconductivity, 2nd ed., McGraw-Hill, 1996.

Tentative Course Content:

Selected topics from chapters 1 – 4.

SC: Super Conductivity

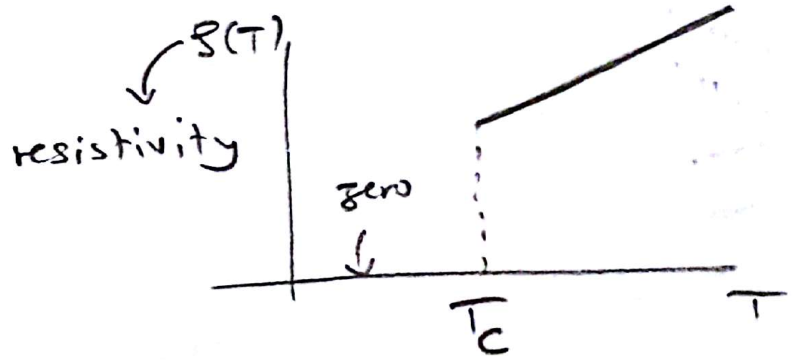
Chapter 1

Historical Overview

* 1911 Kamerlingh Onnes liquifies He and discovers SC

* 1986 High temp. (High T_c) SC (Bednorz & Müller) (liquid Nitrogen)

* 1950, 1960's theory of SC discovered.



1.1 The Basic Phenomena

A. Perfect Conductivity

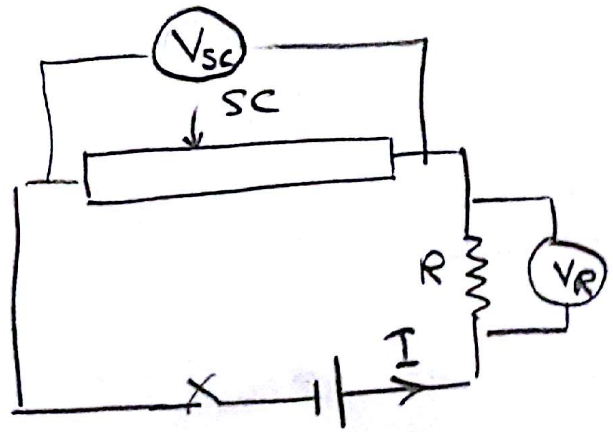
$\vec{E} = -\vec{\nabla} V$

No voltage drop across SC $\Rightarrow$ $\vec{E} = 0$

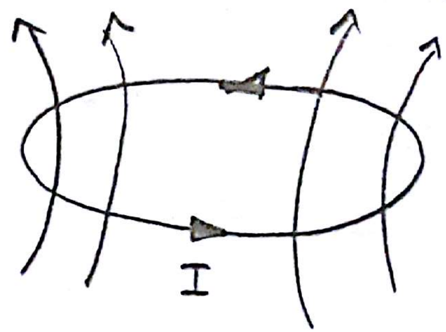
Yet current flows!!

$V_{SC} = 0$

T_c: Critical Temp
Transition Temp

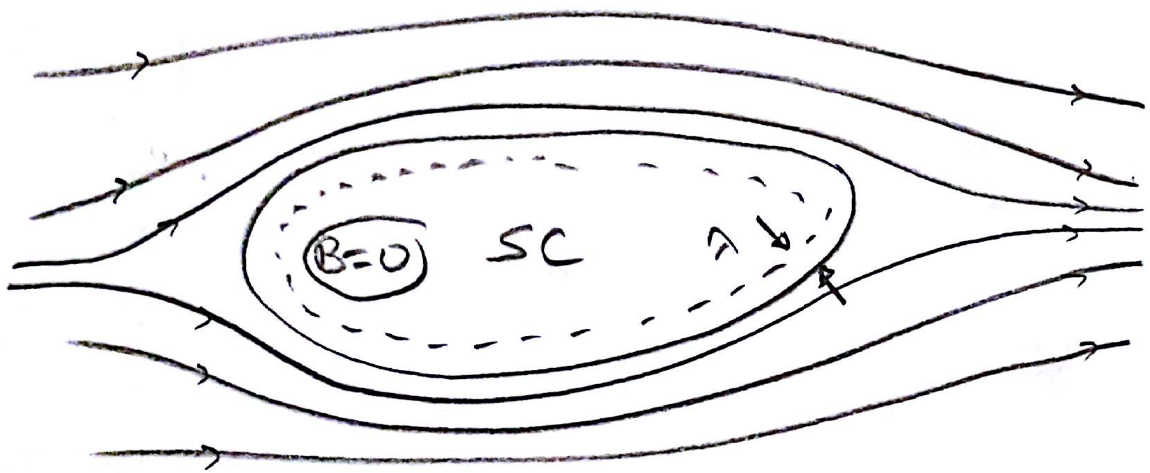


Persistent Currents



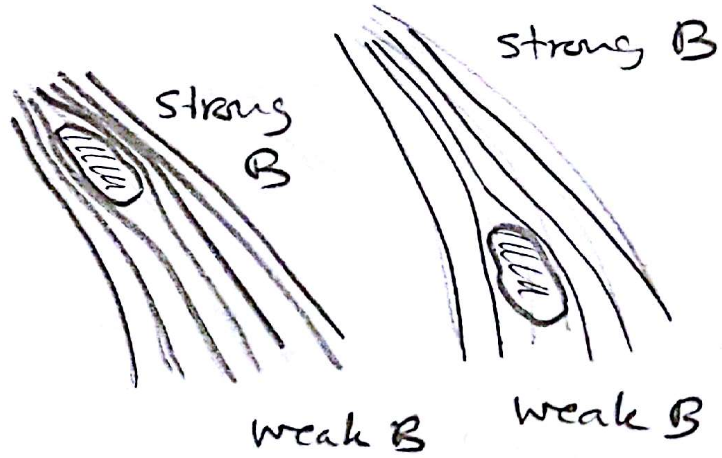
B. Perfect Diamagnetism

1933 Meissner & Ochsenfeld



λ: Penetration depth

- Magnetic Flux excluded from bulk of SC.
- B penetrates in thin layer of thickness λ.
- Sample excludes magnetic field in the SC phase. This raises the magnetic energy, but lowers the overall energy
- Sample is repelled from strong B field ⇒ Diamagnetism



Higher energy

Free energy

System (SC sample + magnetic field inside sample) acts to lower the total free energy :

$$F(T) = U(T) - T\,S(T)$$

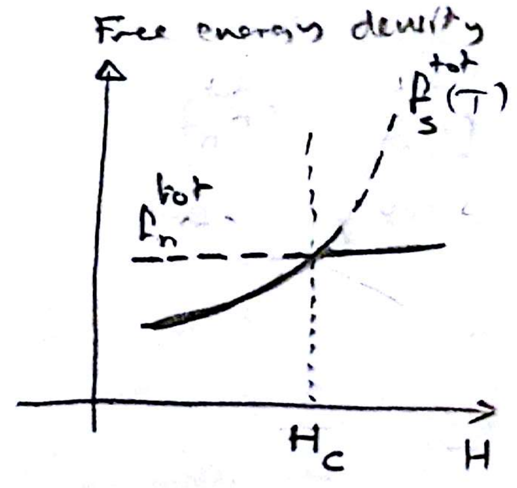
U : Total internal energy (including magnetic energy)

$S(T)$: entropy at temperature T ← abs temp. (K)

$$f(T) = \frac{F(T)}{V} = \text{free energy density} \quad (V\text{: sample volume})$$

$$f_s^{tot}(T) = f_s(T)\Big|_{H=0} + \frac{H^2}{8\pi} \quad (\text{SC state})\ T < T_c$$

$$f_n^{tot}(T) = f_n(T)\Big|_{H=0} \quad (\text{Normal state})\ T > T_c$$

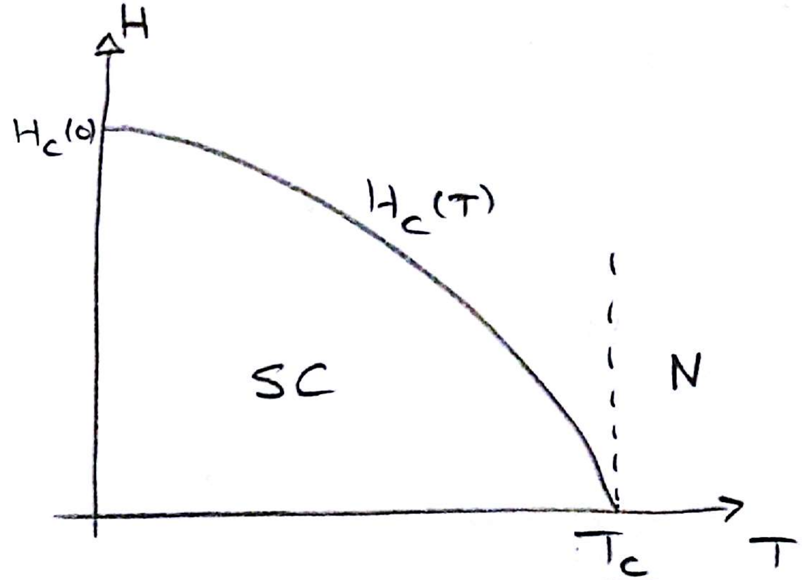


At $H = H_c$, $f_s^{total}(T_c) = f_n^{total}(T_c)$

$$\therefore \quad f_s(T_c)\Big|_{H=0} + \frac{H_c^2}{8\pi} = f_n(T_c)\Big|_{H=0}$$

Temperature Dependence of Critical Field

$$H_c(T) \approx H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

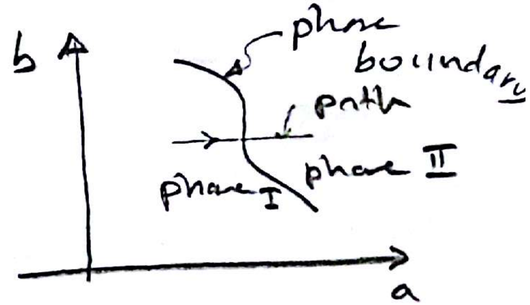


* Superconducting Phase transition:

$H \neq 0$ 1st order

$H = 0$ Second order

* Classification of Phase transitions



$f(a,b)$ a, b : parameters , f : free energy

2nd order transition : All 1st order derivatives $\frac{\partial f}{\partial a}$, $\frac{\partial f}{\partial b}$ phase boundary

1st order transition : Some 1st order derivatives are discontinuous at phase boundary.

The London Equations

1935 F. London & H. London (London brothers)

$$\vec{E} = \frac{\partial}{\partial t}(\Lambda \vec{J}_s) \qquad \Lambda = \frac{m}{n_s e^2} = \frac{4\pi\lambda^2}{c^2}$$

$$\vec{B} = -c\,\vec{\nabla}\times(\Lambda \vec{J}_s)$$

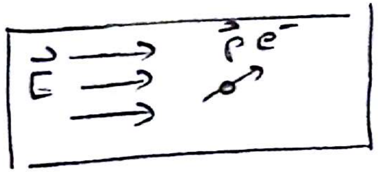
n_s: number density of superconducting e^-

$\vec{J}_s$: supercurrent

λ: penetration depth

* London equations describe perfect conductivity and perfect diamagnetism (i.e. the Meissner effect).

* Perfect Conductivity $\vec{F} = m\vec{a}$



Drude theory: $\vec{F} = \underbrace{-e\vec{E}}_{\text{force from electric field}} - \frac{1}{\tau}\underbrace{\vec{p}}_{m\vec{v}}$ $(\vec{B}=0)$

friction

τ: relaxation time (scattering time)

Put $\tau \to \infty$ (no scattering)

$\therefore \quad \vec{F} = m\vec{a} = -e\vec{E} \Rightarrow m\frac{d\vec{v}}{dt} = -e\vec{E} \qquad \vec{J}_s = -n_s e\vec{v}$

$\Rightarrow \quad \vec{E} = \frac{m}{n_s e^2}\frac{d\vec{J}_s}{dt} = \frac{\partial}{\partial t}(\Lambda \vec{J}_s) \qquad \Lambda = \frac{m}{n_s e^2}$

1st London eqn

Second London Equation

Start with

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$ (Maxwell eqn SI units)

Recall
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$ (" " Gaussian units) → use this

Assume static conditions → $\frac{\partial \vec{E}}{\partial t} = 0$, $\vec{J} = \vec{J}_s$

$\therefore$ $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_s$, Take curl of both sides,

$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}\vec{\nabla}\times\vec{J}_s$, from 2nd eqn, $\vec{\nabla}\times\vec{J}_s = -\frac{1}{c\Lambda}\vec{B}$

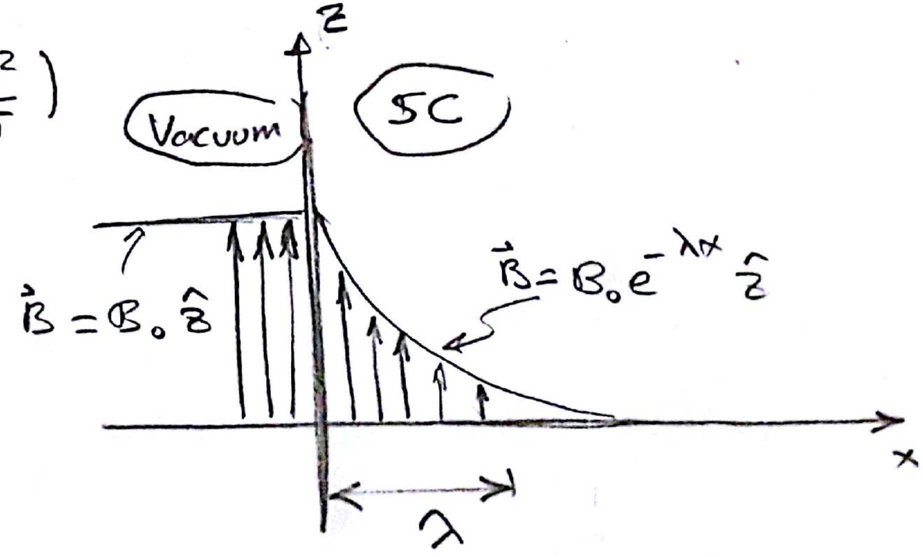
, $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}^2\vec{B}$, $\vec{\nabla}\cdot\vec{B} = 0$

$\therefore$ $-\vec{\nabla}^2\vec{B} = -\frac{4\pi}{\Lambda c^2}\vec{B} = \frac{1}{\lambda^2}\vec{B}$ $\left(\lambda^2 = \frac{\Lambda c^2}{4\pi}\right)$

Assume $\vec{B}(\vec{r}) = B(x)\hat{z}$

$\therefore$ Inside SC : $\frac{d^2B(x)}{dx^2} = \frac{1}{\lambda^2}B(x)$

$\Rightarrow$ $B(x) = Ae^{-\lambda x} + De^{\lambda x}$



Outside SC (in Vacuum) $\vec{J}_s = 0$

$\Rightarrow \vec{\nabla} \times \vec{B} = 0 \Rightarrow \nabla^2 \vec{B} = 0 \Rightarrow \vec{B} = \text{Const} = B_0 \hat{z}$

Boundary Conditions

At $x = 0$ (surface of SC), $\vec{B} = B_0 \hat{z}$ (Meissner effect)

At $x = +\infty$ $\vec{B} = 0$

$\Rightarrow$ reject exponentially increasing solution $(D = 0)$

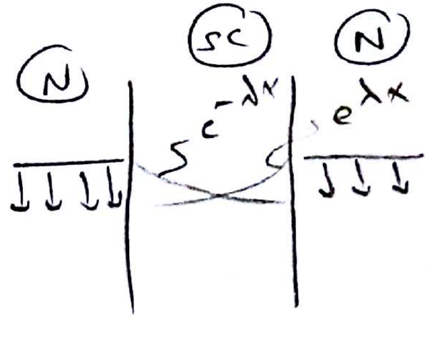
$$\Rightarrow \underbrace{\vec{B} = B_0 \hat{z} \quad x < 0}_{\text{Vacuum}} \Bigg\} \underbrace{\vec{B} = B_0 e^{-\lambda x} \quad x > 0}_{\text{SC}}$$

This is how the 2nd London eqn describe the Meissner effect

* Note

Finite SC slab $(D \neq 0)$.

(Chapter 2, 4)



The electromagnetic Potential $\vec{A}, \varphi$

$\vec{A}(\vec{r},t)$: vector potential

$\varphi(\vec{r},t)$: scalar potential

$$\left.\begin{aligned}\vec{E} &= -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}\right\} \text{Gaussian units.}$$

Coupling to electromagnetic fields in Quantum Mechanics

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + V(\vec{r}) , \qquad \mathcal{H}\psi = i\hbar\frac{\partial}{\partial t}\psi \quad \text{(Time dependent Schrödinger Eqn.)}$$

In the presence of E.M. Potential $(\vec{A}, \varphi)$:

$$\hat{p} \to \hat{p} - \frac{e\vec{A}}{c} , \quad i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - q\varphi$$

Minimal Coupling prescription

* In the absence of fields, kinetic energy is

$KE = \left\langle \frac{\hat{p}^2}{2m} \right\rangle$ and is minimum at $\vec{p} = 0$

* In the presence of fields, $\langle KE \rangle = \left\langle \frac{(\vec{p} - e\vec{A}/c)^2}{2m} \right\rangle$, minimum when $\vec{p} = \frac{e\vec{A}}{c}$

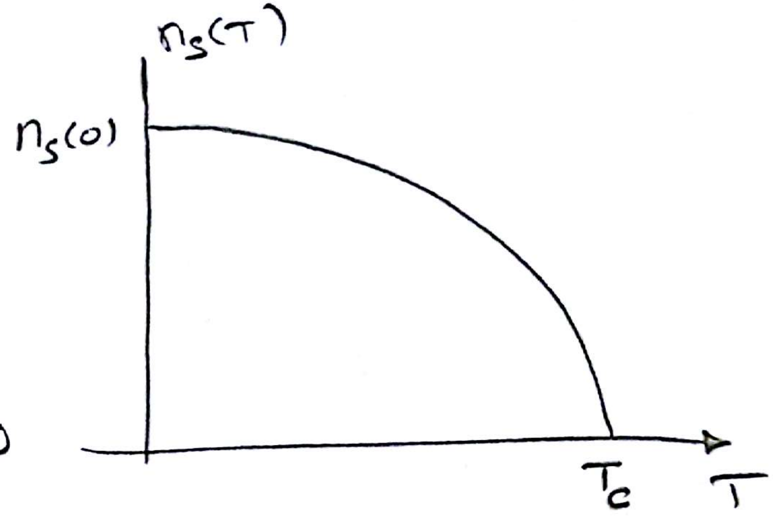
$\therefore \quad \frac{d\vec{p}}{dt} = \frac{e}{c}\frac{\partial \vec{A}}{\partial t}$, since $\vec{p} = m\vec{v}_s = -\frac{m}{n_s e}\vec{J}_s$, $-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}$, we get $\vec{E} = \frac{\partial}{\partial t}(\Lambda \vec{J}_s)$,

like before

The London Penetration Depth $\lambda_L(T)$

From the London equations, we get the penetration depth,

$$\lambda_L(T) = \left(\frac{mc^2}{4\pi\, n_s(T)\, e^2} \right)^{1/2}$$



n_s depends on temperature T.

at $T=0$, if we assume $n_s(0) \simeq \underset{\text{total } e^- \text{ density}}{\underset{\uparrow}{n}}$

Then

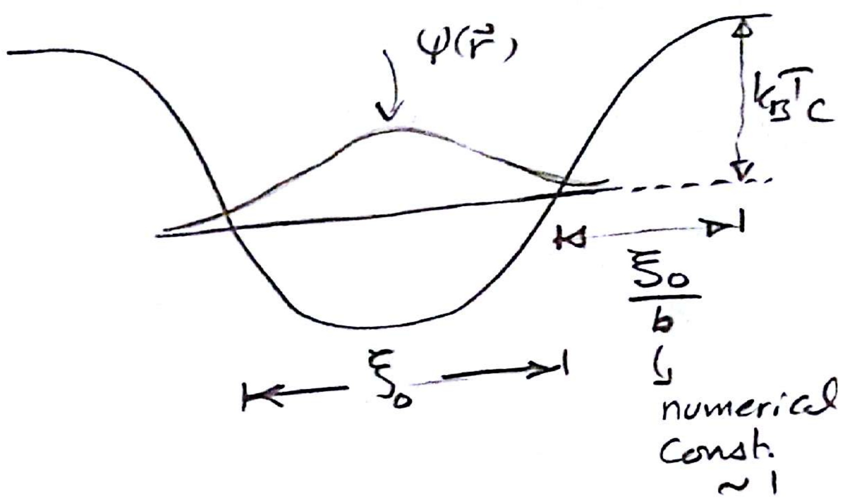
$$\lambda_L(0) \simeq \left(\frac{mc^2}{4\pi n e^2} \right)^{1/2}$$

Actual $\lambda(0)$ tend to be larger than $\lambda_L(0)$.

1.3 The Pippard Non-Local Electrodynamics

The coherence length ξ_0

- What is the approximate size of a Cooper pair?



$$\mathcal{H} = \frac{\hat{P}^2}{2m} + V(\vec{r})$$

$$\langle\psi|\mathcal{H}|\psi\rangle = \langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle + \langle\psi|V(\vec{r})|\psi\rangle$$

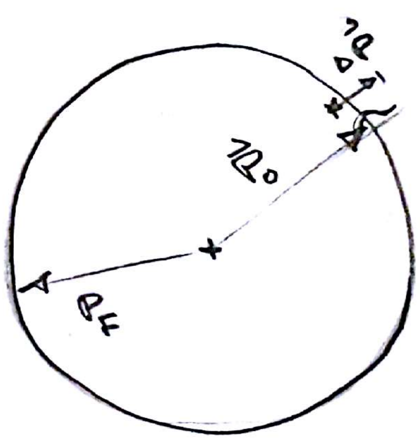
$$\hat{p} = \vec{P}_0 + \Delta\vec{P}$$

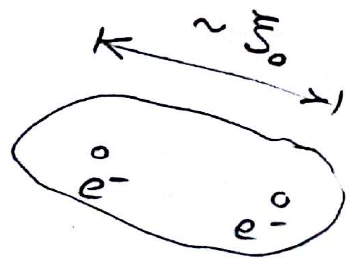
$$\langle\hat{p}^2\rangle_\psi = \langle(\vec{P}_0 + \Delta\vec{P})^2\rangle_\psi$$

$$= \langle\vec{P}_0^2 + 2\vec{P}_0 \cdot \Delta\vec{P} + \underbrace{(\Delta\vec{P})^2}_{\text{ignore}}\rangle_\psi$$

$$= \langle P_F^2 + 2|\vec{P}_0||\Delta\vec{P}|\cos\theta\rangle_\psi$$

$$= P_F^2 + 2P_F\langle|\Delta\vec{P}|\rangle_\psi \, \underbrace{C}_{\text{num. const.} \sim 1}$$





$$\langle\psi|V(\vec{r})|\psi\rangle = \langle\hat{V}(\vec{r})\rangle_\psi$$

$$\therefore \quad E = \frac{P_F^2}{2m} + \frac{P_F\langle\Delta p\rangle}{m}C + \langle V(\vec{r})\rangle$$

All quantities can be estimated in terms of the size ξ_0

$$E(\xi_0) = \frac{p_F^2}{2m} + \frac{p_F}{m}\frac{\hbar}{\xi_0}c + d\,V(\xi_0) \qquad d \sim 1$$

$$\Delta x\,\Delta p \sim \hbar \qquad \Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{\xi_0}$$

Minimize E w.r.t. ξ_0:

$$\frac{dE}{d\xi_0} = 0 = -\frac{\hbar p_F}{m}\frac{c}{\xi_0^2} + d\frac{dV}{d\xi_0}, \qquad \frac{dV}{d\xi_0} \simeq \frac{k_B T_c}{b\,\xi_0}$$

$$p_F = m v_F$$

$$\therefore \quad \hbar v_F \frac{c}{\xi_0^2} \simeq d\,\frac{k_B T_c}{b\,\xi_0}$$

p_F: Fermi momentum

v_F: Fermi velocity

$$\Rightarrow \quad \xi_0 \simeq \left(\frac{bc}{d}\right)\frac{\hbar v_F}{k_B T_c} \qquad \left(\frac{bc}{d}\right) = a \sim 1$$

$$\therefore \quad \xi_0 \simeq a\,\frac{\hbar v_F}{k_B T_c}$$

approximate size of electronic wavepacket

a: number of order unity

- BCS theory gives $a \simeq 0.18$
- Experimentally, $a \simeq 0.15$

Pippard's Formula for supercurrent $\vec{J}_s$

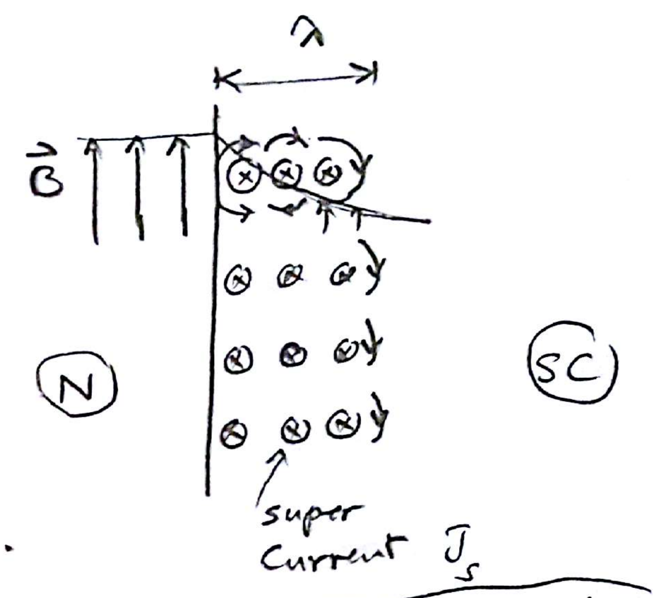
AANLA

LAGAN

$$\vec{J}_s(\vec{r}) = \frac{3}{4\pi \xi_0 \Lambda c} \int \frac{(\vec{r}-\vec{r}')\left[(\vec{r}-\vec{r}')\cdot\vec{A}(\vec{r}')\right]}{|\vec{r}-\vec{r}'|^4} e^{-|\vec{r}-\vec{r}'|/\xi} \, d\vec{r}'$$

$$\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{\ell}$$

ℓ: mean free path



- In Elemental SC (tin, aluminum .. etc.),

$\lambda_L(0) \ll \xi_0$ which gives weakened supercurrent response, i.e. $\vec{J}_s(\vec{r})$ is weakened.

For London theory, $\vec{J}_s = -\frac{n_s e^2}{mc}\vec{A} = -\frac{c}{4\pi\lambda^2}\vec{A}$

$$\lambda_L(0) = \left(\frac{mc^2}{4\pi n e^2}\right)^{1/2}$$

A weakened $\vec{J}_s$ is equivalent to a larger λ than London theory.

$\therefore$ $\lambda > \lambda_L(0)$

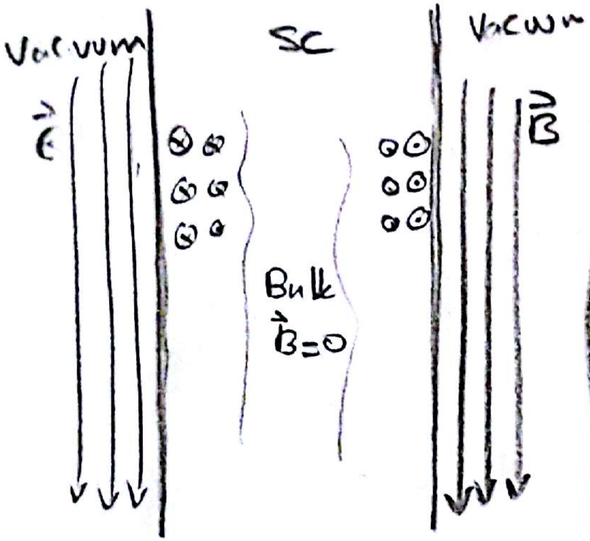


Fig (c)

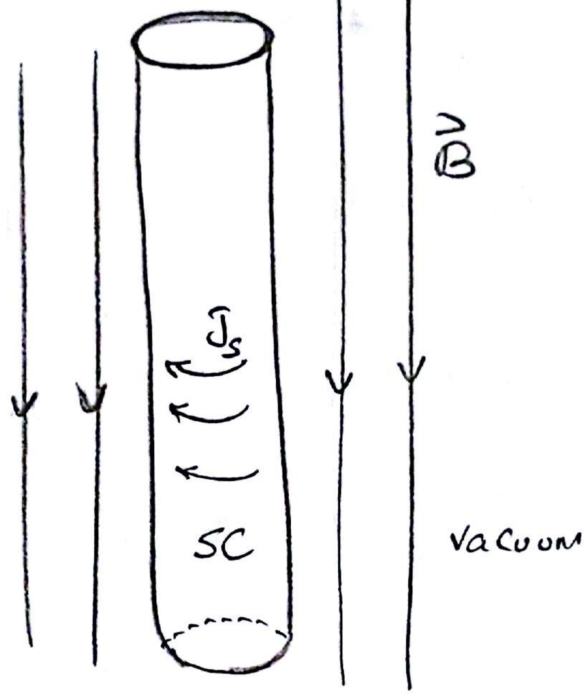
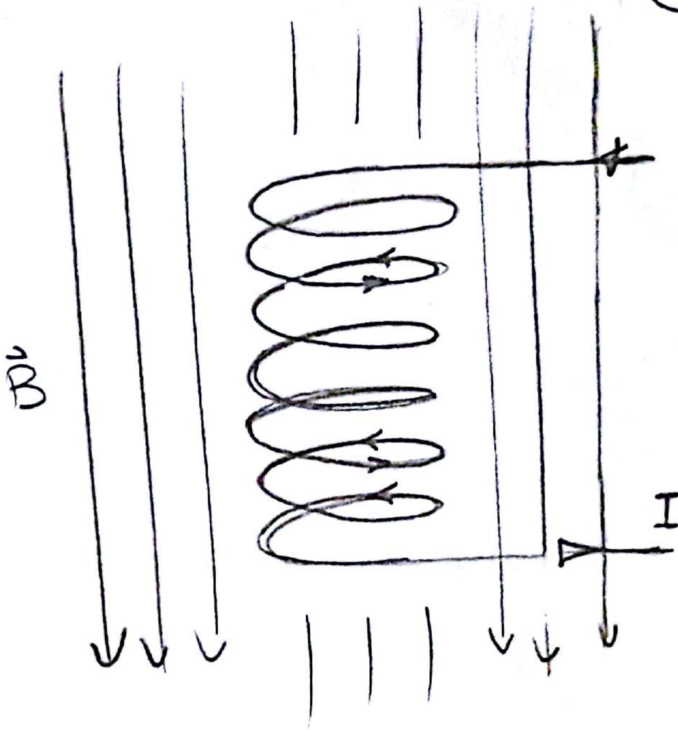


Fig (b)



Fig(a)