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Hamiltonian formalism of GR and numerical relativity - Lecture 3

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Outline

- Lecture 1. Hamiltonian formalism in physics
- Lecture 2. ADM formulation of General Relativity
- Lecture 3. Basics of Numerical Relativity

References

- K. Sundermeyer, "Symmetries in Fundamental Physics" (2014) [Appendix C]
- Eric Gourgoulhon, 3+1 Formalism in General Relativity: Bases of Numerical Relativity (2012) [arXiv:gr-qc/0703035]
- Eric Poisson, A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics (2004)
- Strong Field Gravity (East) - Perimeter Institute for Theoretical Physics course PSI 2018/2019
- T. Baumgarte and S. Shapiro, "Numerical Relativity: Solving Einstein's Equations on the Computer", (2010)

We derived the **Gauss-Codazzi equation**

$$ {}^{(4)}R^{\rho}{}_{\sigma\mu\nu}\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\rho}\gamma^{\gamma}_{\rho}\gamma^{\sigma}_{\delta} = {}^{(3)}R^{\gamma}{}_{\delta\alpha\beta} + K^{\gamma}_{\alpha}K_{\delta\beta} - K^{\gamma}_{\beta}K_{\alpha\delta}. \tag{1} $$

and the **Codazzi-Mainardi equation**

$$ D_{\nu}K_{\mu\rho} - D_{\mu}K_{\nu\rho} = \gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}\gamma^{\gamma}_{\rho}n^{\delta\,(4)}R_{\alpha\beta\gamma\delta}, \tag{2} $$

The constraint equations

- The $3+1$ decomposition of Einstein's equations allows to identify the **intrinsic metric** $\gamma_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ of an initial hypersurface as the **initial data** to be prescribed for the evolution equations of GR.
- We will see that not all the components of γ and K freely propagate, as there are constraints hidden in the Gauss-Codazzi and Codazzi-Mainardi equations. We will assess them by considering the Einstein's equations in vaccuum ${}^{(4)}R_{\mu\nu} = 0$.

The Hamiltonian constraint

- We compute the following contraction of the Gauss-Codazzi equation

$$\gamma^{\alpha\mu}\gamma^{\beta}_{\rho}\gamma^{\nu}_{\sigma}{}^{(4)}R_{\alpha\beta\mu\nu} = {}^{(3)}R_{\rho\sigma} + KK_{\rho\sigma} - K^{\alpha}_{\sigma}K_{\alpha\rho}. \quad (3)$$

- An additional contraction gives

$$\gamma^{\alpha\mu}\gamma^{\beta\nu}{}^{(4)}R_{\alpha\beta\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} \quad (4)$$

- It can be proved that the lhs vanishes, since

$$\begin{aligned} \gamma^{\alpha\mu}\gamma^{\beta\nu}{}^{(4)}R_{\alpha\beta\mu\nu} &= (g^{\alpha\mu} + n^{\alpha}n^{\mu})(g^{\beta\nu} + n^{\beta}n^{\nu}){}^{(4)}R_{\alpha\beta\mu\nu} \quad (5) \\ &= {}^{(4)}R + 2n^{\mu}n^{\nu}{}^{(4)}R_{\mu\nu} + n^{\alpha}n^{\beta}n^{\mu}n^{\nu}{}^{(4)}R_{\alpha\beta\mu\nu} = 0. \quad (6) \end{aligned}$$

- In this way, we get the **Hamiltonian constraint**

$$^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 0. \quad (7)$$

The momentum constraint

- We contract once the Codazzi-Mainardi equation to get

$$D^\nu K_{\mu\nu} - D_\mu K = \gamma^\alpha_\mu \gamma^{\beta\gamma} n^{\delta\,(4)} R_{\alpha\beta\gamma\delta}. \tag{8}$$

- However, we can expand the second γ on the rhs as

$$\begin{aligned} \gamma^\alpha_\mu \gamma^{\beta\gamma} n^{\delta\,(4)} R_{\alpha\beta\gamma\delta} &= -\gamma^\alpha_\mu (g^{\beta\gamma} + n^\beta n^\gamma) n^{\delta\,(4)} R_{\beta\alpha\gamma\delta} && (9) \\ &= -\gamma^\alpha_\mu n^{\delta\,(4)} R_{\alpha\delta} - \gamma^\alpha_\mu n^\beta n^\gamma n^{\delta\,(4)} R_{\alpha\beta\gamma\delta} = 0, && (10) \end{aligned}$$

  where in the last equality we used vacuum Einstein equations and symmetries of the Riemann tensor.
- The final result is the **momentum constraint**

$$D^\nu K_{\mu\nu} - D_\mu K = 0. \tag{11}$$

Evolution equations

- The "Hamiltonian" and "momentum" constraints appear in the Hamiltonian formulation of GR.
- The Hamiltonian and momentum constraints involve only the 3D intrinsic metric $\gamma_{\mu\nu}$, the extrinsic curvature $K_{\mu\nu}$, and their spatial derivatives.
- They are conditions that allow a 3D slice with data $(h_{\mu\nu}, K_{\mu\nu})$ to be embedded in a 4D spacetime $(\mathcal{M}, g_{\mu\nu})$.
- The initial data cannot be freely prescribed due to the existence of the constraint equations.

Evolution equations

- Einstein's equations imply at linear level a wave equation for the components of the metric tensor, which are **second order**.
- We can obtain evolution equations of **first order** by deriving a geometric identity among the Lie derivative of the extrinsic curvature in the direction to the normal of the foliation.
- This is given by the **Ricci equation**

$$\mathcal{L}_n K_{\mu\nu} = n^\delta n^\gamma \gamma^\alpha_\mu \gamma^\beta_\nu {}^{(4)}R_{\delta\gamma\alpha\beta} - \frac{1}{\alpha} D_\mu D_\nu \alpha - K^\rho_\nu K_{\mu\rho}, \qquad (12)$$

which relates the derivative of the K in the normal direction to a hypersurface, to a time projection of the Riemann tensor.

Evolution equations

- In contrast with the 1-form $\omega_\mu = \nabla_\mu t$, we define a **time vector** t^μ such that

$$t^\mu = \alpha n^\mu + \beta^\mu, \quad \beta_\mu n^\mu = 0. \tag{13}$$

It propagates coordinates from one time slice to another, that is, it connects points with the same spatial coordinates.

Evolution equations

- With the previous definition, it can be obtained the evolution equation for the intrinsic metric
$$\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}. \tag{14}$$
- Using the Ricci equation, the Gauss-Codazzi equation, $R_{\mu\nu} = 0$, and some effort, it can be obtained the time evolution for the extrinsic curvature
$$\mathcal{L}_t K_{\mu\nu} = -D_\mu D_\nu \alpha + \alpha \left[{}^{(3)}R - 2K_{\mu\alpha} K^{\alpha}_{\nu} + K K_{\mu\nu} \right] + \mathcal{L}_\beta K_{\alpha\beta} \tag{15}$$
- These are the ADM (Arnowitt-Deser-Misner) equations, which are fully equivalent to the Einstein field equations in vacuum.
- They are first order equations in $(h_{\mu\nu}, K_{\mu\nu})$, but they are a weakly hyperbolic system of partial differential equations, which creates trouble for numeric resolution.
- An improved extension with strong hyperbolicity is contemplated in the BSSNOK formalism.

Lagrangian of GR

- From our previous analysis we can derive the following decomposition

$$^{(4)}R = {}^{(3)}R + K_{ij}K^{ij} - K^2, \tag{16}$$

where $K = K^i{}_i$ and $^{(3)}R$ is the Ricci scalar in 3D. (There are boundary terms that do not affect the field equations, but are important for spacetimes with boundaries or nontrivial boundary conditions).

- We observe that the Ricci scalar in 4D can be decomposed into a **kinetic** term quadratic in K which contains time derivatives of γ_{ij}, and a **potential** term $^{(3)}R$ containing only γ_{ij} and its spatial derivatives.
- In consequence, we can write the Einstein-Hilbert action in 3+1 form as

$$S_E = \frac{1}{2\kappa}\int d^4x\sqrt{\gamma}\alpha(K_{ij}K^{ij} - K^2 + {}^{(3)}R). \tag{17}$$

Primary constraints in GR

- The time derivatives of α and β^i do not appear in the action. Therefore, we can safely recognize the following primary constraints

$$\pi_0 = \frac{\delta L}{\delta \dot{\alpha}} = 0, \qquad \pi_i = \frac{\delta L}{\delta \dot{\beta}^i} = 0. \tag{18}$$

- These are all the primary constraints, since in all the remaining momenta

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\alpha}\frac{\delta L}{\delta K_{ij}} = \frac{\sqrt{\gamma}}{2\kappa}(K^{ij} - K\gamma^{ij}), \tag{19}$$

all the velocities can be solved in terms of the momenta as

$$\dot{\gamma}_{ij} = \frac{2\kappa}{\sqrt{\gamma}}(2\pi_{ij} - \pi^k_k \gamma_{ij}) + 2D_{(i}\beta_{j)}. \tag{20}$$

- (Remember that)

$$K_{ij} = \frac{1}{\alpha}\left(-\frac{1}{2}\dot{\gamma}_{ij} + D_{(i}\beta_{j)}\right). \tag{21}$$

Hamiltonian

- Therefore, we can write the gravitational Hamiltonian

$$H = \int d^3x(\dot{\gamma}_{ij}\pi^{ij} - L + \lambda_0\pi_0 + \lambda^i\pi_i) \tag{22}$$

- Replacing the 3+1 Lagrangian and the velocities in terms of the momenta, it is obtained

$$H = \int d^3x \left[\frac{2\kappa\alpha}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^i_i)^2\right) + 2\pi^{ij}D_iN_j - \frac{\alpha\sqrt{\gamma}}{2\kappa}{}^{(3)}R\right] + \int d^3x(\lambda_0\pi_0 + \lambda^i\pi_i) \tag{23}$$

- After obtaining the Hamiltonian, we need to make sure our primary constraints are preserved over time.

Poisson brackets in GR

- The Poisson brackets in GR are defined as

$$
\begin{aligned}
\{\gamma_{ij}(x), \gamma_{kl}(y)\} &= 0, \\
\{\pi^{ij}(x), \pi^{kl}(y)\} &= 0, \\
\{\gamma_{ij}(x), \pi^{kl}(y)\} &= \frac{1}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l)\delta^{(3)}(x-y).
\end{aligned}
\tag{24}
$$

- Note that the PB work this way when indices are in "canonical" positions. Otherwise, some variational properties are needed.
- When computing spatial derivatives of the fundamental variables, care must be taken with derivatives of Dirac delta, or used "smeared" constraints.

Time evolution of primary constraints

- The consistency of primary constraints imply the presence of secondary constraints. Firstly,

$$\dot{\pi}_0 = \{\pi_0, H\} = -\frac{2\kappa}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i_i)^2\right) + \frac{\sqrt{\gamma}}{2\kappa}{}^{(3)}R \equiv \mathcal{C}_0. \quad (25)$$

  To ensure consistency, we must impose that this expression is weakly zero. This is the **Hamiltonian constraint**.

- An alternative form for it is

$$\mathcal{C}_0 \equiv -2\kappa G_{ijkl}\pi^{ij}\pi^{kl} + \frac{1}{2\kappa}\sqrt{\gamma}{}^{(3)}R = 0, \quad (26)$$

  where the so-called supermetric is defined as

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}). \quad (27)$$

Momenta constraints

- The time evolution of the remaining primary constraints can be written as

$$\dot{\pi}_i = \{\pi_i, H\} = 2\sqrt{\det(\gamma)}D^j(\det(\gamma)^{-1/2}\pi_{ij}) = 2D_j\pi_i^j \equiv \mathcal{C}_i. \quad (28)$$

- Imposing that this expression is weakly zero, it is obtained the **momenta constraints**.
- A boundary term $2\int d^3xD_i(\pi^{ij}\beta_j/\sqrt{\det\gamma})$ has been ignored for this derivation.
- Time evolution of $\mathcal{C}_0$ and $\mathcal{C}_i$ does not give new constraints, therefore no more secondary constraints appear and Dirac's algorithm is finished.
- The total Hamiltonian is

$$H = \int d^3x(\alpha\mathcal{C}_0 + \beta^i\mathcal{C}_i + \lambda_0\pi_0 + \lambda_i\pi_i) \quad (29)$$

and it is a linear combination of constraints. The parameters of the combination are Lagrange multipliers.

The ADM algebra

- The expressions containing the information about a closed algebra are

$$
\begin{aligned}
\{\mathcal{C}_i(x), \mathcal{C}_j(y)\} &= -\mathcal{C}_j(x)\partial_i^y \delta(x,y) + \mathcal{C}_i(y)\partial_j^x \delta(x,y), \\
\{\mathcal{C}_i(x), \mathcal{C}_0(y)\} &= \mathcal{C}(x)\partial_i^x \delta(x,y), \\
\{\mathcal{C}_0(x), \mathcal{C}_0(y)\} &= \gamma^{ij}(x)\mathcal{C}_i(x)\partial_j^y \delta(x,y) - \gamma^{ij}(y)\mathcal{C}_i(y)\delta_j^x(x,y).
\end{aligned}
\tag{30}
$$

- Schematically,

$$
\begin{aligned}
\{\text{momenta}, \text{momenta}\} &= \text{momenta} \\
\{\text{momenta}, \text{Hamiltonian}\} &= \text{Hamiltonian} \\
\{\text{Hamiltonian}, \text{Hamiltonian}\} &= \text{momenta}.
\end{aligned}
\tag{31}
$$

- All Poisson brackets among Hamiltonian and momenta constraints give as a result a combination of themselves. Since they weakly vanish on the constraint surface, the PB also weakly vanishes, and they are preserved in time.

Counting of degrees of freedom

The counting of degrees of freedom in GR goes as follows:

- The pairs of canonical variables (γ_{ij}, π^{ij}), which are symmetric in the $i-j$ indices, give $4 \cdot (4-1)/2 = 6$ degrees of freedom (since $i, j = 1, .., 3$).
- We remove 4 degrees of freedom with the 4 first class constraints $\mathcal{H}_\mu$
- It remains $4 \cdot (4-3)/2 = 2$ degrees of freedom of GR.
- These are the two degrees of freedom associated with the two possible polarizations of gravitational waves.

Hamilton's equations for GR

Hamilton's equations for GR are written as

$$\dot{g}_{ij} = \{g_{ij}, H\} = \frac{\delta H}{\delta \pi^{ij}}, \qquad \dot{\pi}^{ij} = \{\pi^{ij}, H\} = -\frac{\delta H}{\delta g_{ij}}. \tag{32}$$

From the Hamiltonian for GR previously found, it is obtained after some computations that

$$\dot{g}_{ij} = 2\alpha g^{-1/2}(\pi_{ij} - \frac{1}{2}g_{ij}\pi) + \beta_{i|j} + \beta_{j|i}, \tag{33}$$

$$\begin{aligned}\dot{\pi}^{ij} = & -\alpha\sqrt{g}({}^{(3)}R_{ij} - \frac{1}{2}g^{ij\,(3)}R) + \frac{1}{2}\alpha g^{-1/2}g^{ij}(\pi^{kl}\pi_{kl} - \frac{1}{2}\pi^2) \\ & - 2\alpha g^{-1/2}(\pi^{ik}\pi_k{}^j - \frac{1}{2}\pi\pi^{ij}) + \sqrt{g}(\beta^{|ij} - g^{ij}\beta^{|k}{}_{|k}) \\ & + (\pi^{ij}\beta^k)_{|k} - \beta^i_{|k}\pi^{kj} - \beta^j{}_{|k}\pi^{ki},\end{aligned} \tag{34}$$

where $_{|i}$ is short hand for covariant derivative wrt i.

General relativity in ADM formalism

The ADM set of equations can be casted in first-order derivatives system of equations of the variables a_i, d_{ijk} and K_{ij}

$$\partial_0 a_i \simeq -\alpha \partial_i Q, \tag{35}$$

$$\partial_0 d_{ijk} \simeq -\alpha \partial_i K_{jk}, \tag{36}$$

$$\partial_0 K_{ij} \simeq -\alpha \partial_k \Lambda^k_{ij}, \tag{37}$$

where $a_i = \partial_i ln\alpha$, $d_{ijk} = \frac{1}{2}\partial_i \gamma_{jk}$, Q depending on derivatives of lapse, and Λ^k_{ij} depending on d_{ijk} and a_i.
It can be proved that this system is weakly hyperbolic, except for a very specific type of initial data [Alcubierre (2008)] .

Consider a first order system of evolution equations of the form

$$\partial_t u + M^i \partial_i u = s(u), \tag{38}$$

where M^i are $n \times n$ matrices, and $s(u)$ a source vector (set to zero). M^i are called *characteristic matrices*. By building the principal symbol matrix $P(n_i) = M^i n_i$ (n_i arbitrary unit vector), then the system is

- strongly hyperbolic, if the principal symbol has real eigenvalues and a complete set of eigenvectors for all n_i, and
- weakly hyperbolic, if P has real eigenvalues for all n_i, but does not have a complete set of eigenvectors.

General relativity in BSSNOK formalism

- The most widely used formulation in three-dimensional numerical codes based on the 3+1 decomposition, is the BSSNOK one (Nakamura, Oohara, Kojima (1987), Shibata, Nakamura (1995), Baumgarte, Shapiro (1998)).
- It has proven to be very robust in practice in a large class of systems with strong and dynamical gravitational fields, with and without matter.

General relativity in BSSNOK formalism

- It is considered a conformal rescaling of the spatial metric

$$\tilde{\gamma}_{ij} = \phi^{-4}\gamma_{ij}, \tag{39}$$

  in such a way that the conformal metric $\tilde{\gamma}_{ij}$ has unit determinant $\phi^4 = \gamma^{1/3}$.
- The extrinsic curvature is decomposed into its trace K and traceless part A_{ij}, and conformally transform it $A_{ij} = e^{4\phi}\tilde{A}_{ij}$ is postulated

$$K_{ij} = e^{4\phi}\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{40}$$

BSSNOK formalism

In terms of these variables the Hamiltonian constraint becomes

$$H = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 = 0, \quad (41)$$

while the momentum constraint becomes

$$\bar{D}_j(e^{6\phi}\tilde{A}_{ij}) - \frac{2}{3}e^{6\phi}\bar{D}^i K = 0. \quad (42)$$

BSSNOK equations

The evolution equation for γ_{ij} splits into two equations

$$\partial_t \phi = -\frac{1}{6}\alpha K + \beta^i \partial_i \phi + \frac{1}{6}\partial_i \beta^i, \tag{43}$$

$$\partial_t \bar{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \beta^k \partial_k \bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j \beta^k + \bar{\gamma}_{kj}\partial_i \beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k \beta^k. \tag{44}$$

BSSNOK equations

The evolution equation for K_{ij} splits into the two equations

$$\partial_t K = -\gamma^{ij} D_j D_i \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) + \beta^i \partial_i K, \tag{45}$$

$$\begin{aligned}\partial_t \tilde{A}_{ij} =& e^{-4\phi}(-D_i D_j \alpha + \alpha R_{ij})^{TF} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) \\ &+ \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k.\end{aligned} \tag{46}$$

In the last equation we have denoted the trace-free part of a tensor T_{ij} as

$$T_{ij}^{TF} = T_{ij} - \gamma_{ij} T^k_k / 3.$$

We also split the Ricci tensor into $R_{ij} = \bar{R}_{ij} + R^{\phi}_{ij}$. The piece $\bar{R}_{ij}$ is expressed in terms of the conformal connection functions

$$\bar{\Gamma}^i = \bar{\gamma}^{jk}\bar{\Gamma}^i{}_{jk} = -\partial_j \bar{\gamma}^{ij}, \tag{47}$$

which yields

$$\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj}). \tag{48}$$

Then, the $\bar{\Gamma}^i$ are treated as independent functions that satisfy their own evolution equations, which are

$$\begin{aligned}\partial_t\bar{\Gamma}^i = &-2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K + 6\tilde{A}^{ij}\partial_j\phi\right)\\ &+ \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j + \bar{\gamma}^{lj}\partial_j\partial_l\beta^i.\end{aligned} \tag{49}$$

Overview of numerical relativity

- General Relativity remains largely untested, except in weak-field, slow-velocity regime. Solutions, except for cases with high symmetry, have not been obtained for important dynamical scenarios thought to occur in nature.
- With the advent of supercomputers, it is possible to start tackling these scenarios in detail, that is, high-velocity, strong-field regimes.
- Among others, it comprehends gravitational collapse to black holes and neutron stars, inspiral and coalescence of binary black holes and neutron stars, and generation and propagation of gravitational waves.

Overview of numerical relativity

- The equations arising in numerical relativity are multidimensional, nonlinear, coupled partial differential equations in space and time. (like fluid dynamics, magnetohydrodynamics, and aerodynamics)
- However, GR has unique additional complications. First is the choice of coordinates. Often some choice of coordinates turns out to be bad, since singularities appear in the equations.
- Another complication comes from calculation of waveforms from astrophysical sources of gravitational radiation. Extracting the waves from the background of a simulation requires to prove the numerical spacetime in the far-field, which is usually very distant in space and might require lots of computational resources.