Dyonic Taub-NUT Phases

I - TN Thermodynamics:

i) Background:

* Forms & Conservation laws
* Dirac Monopole

ii) Taub-NUT space:

* Non-relativistic weak field limit
* Misner string

iii) TN Thermodynamics:

* Unconstrained Thermodynamics
* Conserved charges of TN spaces
* Thermodynamics Ensembles and Regularity conditions
* Our approach to TN Thermodynamics

Forms & Conservation Laws

* A n-form is a fully anti-sym. n-rank tensor, or
  $w_{[\mu_1 \cdots \mu_n]}$

Also written as

$$w = \omega_{\mu_1 \cdots \mu_n} \, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}.$$

* Exterior derivative "d" which takes n-form $\longrightarrow$ (n+1)-form,

$$dw = (n+1) \nabla_{[\mu_1} \omega_{\mu_2 \cdots \mu_{n+1}]} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n+1}}.$$

* $ddw = 0$, Since $\nabla_{[\mu} \nabla_\nu \, \omega_{\mu_1 \cdots \mu_n]} = 0.$

* Hodge operation $(*)$ on an n-form $\longmapsto$ $(d-n)$-form ($d = \dim$)

$$(* w)_{\mu_{n+1} \cdots d} = \frac{1}{n!} \epsilon^{\mu_1 \cdots \mu_n}{}_{\mu_{n+1} \cdots \mu_d} \, \omega_{\mu_1 \cdots \mu_n}.$$

Examples: EM vector Pot. $A_\mu(\varphi, \vec{A}) \equiv A$ (1-form)

$$dA = 2 \nabla_{[\mu} A_{\nu]} \, dx^\mu \wedge dx^\nu \longrightarrow \text{2-form}.$$

Maxwell's Eqn's

$$\begin{cases} \nabla_\mu F^{\nu\mu} = J^\nu \Rightarrow d(*F) = {}^*J \\ \nabla_{[\mu} F_{\nu\alpha]} = 0 \Rightarrow dF = 0. \end{cases}$$

<u>Stock's Theorem:</u>

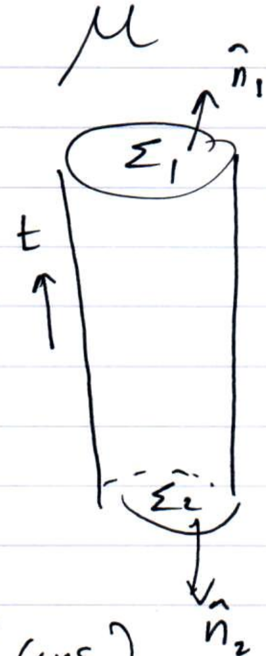
* A Conserved Current $\nabla_\mu J^\mu = 0 \Longleftrightarrow d(*J) = 0.$

$$(*J)_{\alpha\beta\gamma} = \epsilon^\mu{}_{\alpha\beta\gamma} J_\mu \Rightarrow (d*J)_{\alpha\beta\gamma\delta} \propto \epsilon_{\alpha\beta\gamma\delta} \, f$$

$(\nabla_\mu J^\mu)$

$$\therefore \; d(*J) = \epsilon \, \nabla_\mu J^\mu = \epsilon_{\alpha\beta\gamma\delta} (\nabla_\mu J^\mu) \, dx^\alpha \wedge \cdots \wedge dx^\delta$$
$$= \epsilon_{0123} \nabla_\mu J^\mu \, dx^0 \wedge \cdots \wedge dx^3$$
$$\int d(*J) = \int \sqrt{-g} \, \nabla_\mu J^\mu \, d^4x$$

Statement of stock's Theorem:



$$\int_{\mathcal{M}} d(*J) = \int_{\partial\mathcal{M}} *J$$

Similar to vector calculus Divergence's theorem

$$\int_V \bar{\nabla}\cdot\bar{A}\, dV = \oint_S \bar{A}\cdot d\bar{S}$$

Notice:

$\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$.

$$\int_{\partial\mathcal{M}} *J = \underbrace{\int_{\Sigma_1} *J}_{Q_1} - \underbrace{\int_{\Sigma_2} *J}_{Q_2} \Rightarrow Q_1 = Q_2 \text{ (cons.)}$$

$\therefore$

$$Q_e = \int_{\Sigma} *J = \int_{\Sigma} d*F = \int_{\partial\Sigma} *F$$

Also if $dF = -*J_m \Rightarrow \int_{\Sigma} dF = \int_{\partial\Sigma} F = -\int_{\Sigma} *J_m = Q_m$

$\therefore$ $Q_m = -\int_{\partial\Sigma} F$

$$Q_e = \int_{\partial\Sigma} *F$$

Notice:

If $F \longrightarrow *F$, $*F \longrightarrow **F = -F$, one gets

$$\tilde{Q}_e = \int_{\partial\Sigma} **F = -\int_{\partial\Sigma} F = Q_m$$

$$\tilde{Q}_m = -\int_{\partial\Sigma} *F = -Q_e$$

$\therefore$ $Q_e \longrightarrow Q_m$ & $Q_m \longrightarrow -Q_e$ (EM duality!)

* In GR we have a similar expression for obtaining total mass and angular momentum of a spacetime. These quantities are associated with spacetime isometry sym. which is connected to the existence of Killing vectors, or

$$\nabla^\mu \chi^\nu + \nabla^\nu \chi^\mu = 0.$$

+ $\chi^{\mu(t)} = (1,0,0,0)$ (time-translat. sym.)

+ $\chi^{\mu(\varphi)} = (0,0,0,1)$ (rotat. sym.)

$$Q_{(\chi)} = -c \int_{\partial\Sigma^\infty} (* d\chi) = \begin{cases} M \text{ (mass), for } \chi = \chi^{(t)} \\ J \text{ (Ang. mom. for } \chi = \chi^{(\varphi)} \end{cases}$$

* Notice the similarity with electric charge ξ. and mass!

$$M = -c \int_{\partial\Sigma^\infty} (* d\chi^{(t)})$$

Do we have a dual quant for mass? Yes it is the nut charge (parameter)

$$\tilde{M} = \int_{\partial\Sigma^\infty} (d\chi^{(t)}) = n. \quad \text{(in Minkowski space)}$$

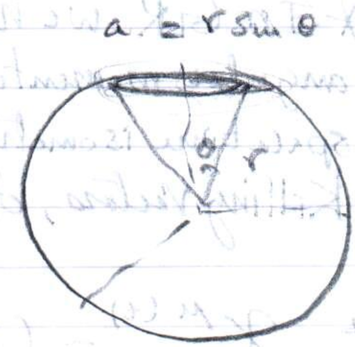
* "n" is interpreted as a source for a magnetic-type mass which is related to gravitomagnetic phenomena!

To understand the nature of "n" and Taub-NUT metric it is instructive to go over the topic of Dirac' Monopole!

$$d\ell = \overset{a}{\overbrace{r\sin\theta_0}}\, d\varphi$$

πa^2

$$\oint_{S^2_\infty} \bar{B}\cdot d\bar{S} = \lim_{a\to 0} \int_{S^2 - \mathrm{disk}(\omega)} \bar{B}\cdot d\bar{S}$$



$$= \oint_{C_{-a}} \bar{A}\cdot d\bar{\ell} = \oint A_\varphi\, a\, d\varphi = 4\pi P$$

$$A_\varphi \sim \frac{\# P}{a} \Longrightarrow \frac{\# P}{r\sin\theta_0}$$

i.e. A_φ is not well defined along z axis!

Dirac Monopole:

forcing the existence of magnetic monopole charge led Dirac to

$$\bar\nabla \cdot \bar B = 4\pi p \,\delta \quad \bar B = \frac{P}{r^2}\,\hat e_r .$$

$$\oint \bar B \cdot d\bar A = 4\pi p \qquad \bar B \overset{?}{=} \bar\nabla \times \bar A .$$

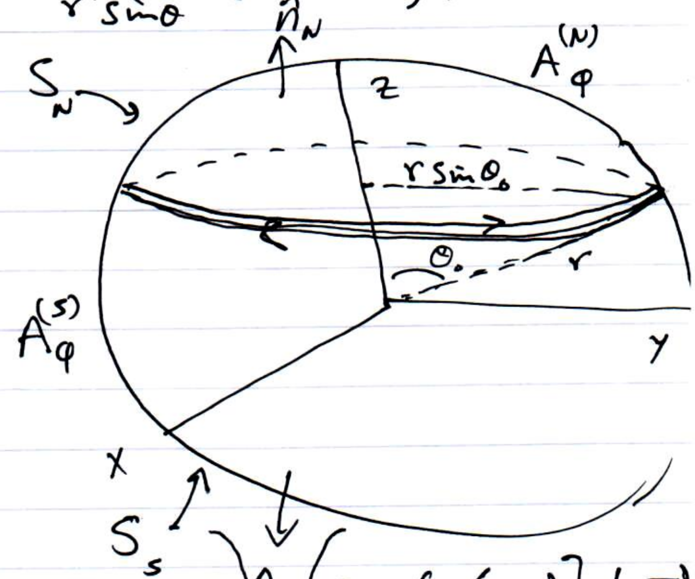
* In fact we can write $\bar B = \bar\nabla \times \bar A$, but $\bar A$ can not be well defined globally, or

$$A^\varphi = \frac{P}{r\sin\theta}\,(\cos\theta + C).$$

* Notice that, when $C=0$, A^φ is not well defined along z-axis.

* But one can have a well-defined A using two patches?

$$A^\varphi = \begin{cases} A^\varphi_{(N)} = \dfrac{P}{r\sin\theta}(\cos\theta - 1), & z>0. \\[2ex] A^\varphi_{(S)} = \dfrac{P}{r\sin\theta}(\cos\theta + 1), & z<0. \end{cases}$$



$$P = \frac{1}{4\pi}\oint \bar B \cdot d\bar s$$

$$= \frac{1}{4\pi}\left[\int_{S_N} \bar B_N \cdot d\bar s + \int \bar B_s \cdot d\bar s\right]$$

$$= \frac{1}{4\pi}\left[\oint_{C_N} \bar A_N \cdot d\bar r + \oint_{C_s} \bar A_s \cdot d\bar r\right]$$

$$= \frac{r_0\sin\theta_0}{4\pi}\left[\oint_{C_N} A^N_\varphi\, d\varphi - \int_{C_s} A^s_\varphi\, d\varphi\right] = P.$$

$$= \frac{r\sin\theta_0}{4\pi}\left[\frac{P}{r\sin\theta_0}(\cos\theta_0 + 1) - \frac{P}{r\sin\theta_0}(\cos\theta_0 - 1)\right](2\pi)$$

Taub-NUT metric in Minkowski space:

* $ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + f^{-1}dr^2 + (r^2+n^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$f = \dfrac{r^2 - n^2 - 2Mr}{r^2+n^2}$, M is mass parameter, n is nut parameter.

* Notice:
- Metric goes to schw. as $n \to 0$.
- Metric does not go to Minkowskin as $m \to 0$!

* To understand this metric one better go to Newtonian Limit
In this limit $g_{\mu\nu} \triangleq \eta_{\mu\nu} + h_{\mu\nu}$ (small) for big mass M.

$\dfrac{dx^i}{dt} \ll 1$, for test mass m!

* For the large mass M, one gets
the $G_{\mu\nu} = K T_{\mu\nu} \Rightarrow \nabla^2\varphi = 0$
for small mass $t = \tau$

$\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{d\tau}\dfrac{dx^\beta}{d\tau} = 0 \Rightarrow m\dfrac{d^2x^i}{dt^2} = m\,\partial_i\phi$

$\Rightarrow \phi = -\dfrac{M}{r} \Rightarrow S_p \sim \int dt\, (\tfrac{1}{2}mv^2 - m\phi)$.

* Perturbations can be decomposed

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; $h_{\mu\nu} = \begin{cases} h_{00} \to \text{scalar}(1) \\ h_{0i} \to \text{Vector}(3) \\ h_{ij} \to \text{tensor}(6) \end{cases}$

* GR predicts the possibility of deviating from Newtonian gravity in weak field limit / non-relativistic limit if $h_{0i} = A_i$ is non-vanishing asympt. as $r \to \infty$. This is what happens in Taub-NUT space

As $r \to \infty$ $\quad ds^2 = -dt^2 + r^2 d\Omega_2^2 - 4n\cos\theta\, dt\, d\varphi$.

$h_{0i} = A_\varphi = -2n\cos\theta$.

Doing the weak field + non-relativistic limit

$$h_{00} = -2\phi_G \,, \quad \phi = -\frac{M}{r} \quad \& \quad h_{0i} = A_i \,, \quad A_\varphi = -2n\cos\theta .$$

In this case the E.O.M of a test particle with mass "m" is

$$m\frac{d^2x^i}{dt^2} = -m\,\partial_i \phi_G + \frac{m}{c}\left[\bar{v}\times(\bar{\nabla}\times\bar{A}_G)\right]_i$$

with

$$\boxed{\bar{a} = -\frac{M}{r^2}\hat{r} - \frac{2nc}{r^2}\left(v_\varphi\hat{\theta} - v_\theta\hat{\varphi}\right)}$$

$$|\bar{a}| = \sqrt{M^2 + 4n^2c^2(v_\theta^2 + v_\varphi^2)}\,/\,r^2 .$$

with Lagrangian

$$S_p \sim \int dt\left(\tfrac{1}{2}mv^2 - m\phi_G + \frac{m}{c}\bar{A}_G\cdot\bar{v}\right)$$

which is similar to a particle under the influence of Lorentz force

$$S_p \sim \int dt\left(\tfrac{1}{2}mv^2 - e\phi_e + \frac{e}{c}\bar{A}_e\cdot\bar{v}\right).$$

A_i is the source of gravitomagnetic field, with the charge "n"

$$n = \int_{\partial\Sigma^\infty} dx^{(t)} \qquad (\text{dual mass!})$$

* Therefore, TN is interpreted as a gravitational dyone!

* Also, n is automatically conserved!